PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, generic, discipline specific, elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade "A".

UGC (Open and Distance Learning Programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Material (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

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Vice-Chancellor

Netaji Subhas Open University
Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Physics (HPH)
Course : Electricity & Magnetism
Course Code : CC-PH-08

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NETAJI SUBHAS
OPEN UNIVERSITY

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UNIT 1 : Electric Field and Electric Potential

1.1 Objective

After completing this unit you will be able to understand

1. Electrostatic interaction between charges through Coulomb's law.
2. Electric field conception to explain the propagation of interaction by introducing field lines conception.
3. A vector presentation of electric field through the introduction of electric field intensity conception-a vector representation of electric field in space.
4. Electric flux, Gauss's theorem and its application.
5. Presentation of electric property by a scalar field conception through introduction of electric potential.
6. Presentation of electric field intensity $\vec{E}$ as a gradient of electric potential V. Equipotential surfaces.

7. Conservative nature of electric field, Laplace's and Possion's Equations.
8. Energy assoicated with a symmertic charge distribution.
9. Capacitance of capacitors.
10. Electrical image and its application to some specific cases.

1.2 Introduction

The term 'electricity' started its path from the experiment of Thales (600 BC)-Greek philosopher who rubbed Amber with silk and it was seen both of them developed the property of attracting small papers bits. As the Greeks called Amber as electron, so the term electricity boiled down. Electricity was in its rudimentary state still late 18^{th} century until, about 100 years after the introduction of Newton's Law of Gravitation (1687), Coulomb in 1785 AD, introduced the law governing the interaction between the charges – the subject electricity got its space.

Atom, the basic ingredient of matter contains two charged particles called electron (negatively charged) and proton (positively charged) carrying equal but opposite charges of magnitude 1.6×10^{19} Coulomb each, which is the smallest quantum of charge that can exist in nature. Obviously,

Any charge that physically exists will be the integral multiple

of the smallest quantum of charge - the magnitude of the

charge of an electron, this is known as ***quantisation*** *of charge.*

The charge also follows another law called conservation of charge which goes as,

The total charge of an isolated system remains ***conserved.***

but this conservation is not like mass conservation law which changes with the speed of reference frame. Charge

conservation law is independent of reference frame

1.3 Electrostatics in Vacuum

(a) Coulomb's Law and Electric Field :

Coulomb's law gives the interaction between the two static point charges. The law states that the force of interaction between two point charges separated by a distance is,

i) directly proportional to the product of the charges,

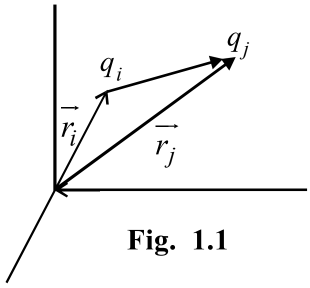


Fig. 1.1

ii) inversely proportional to the square of the distance of separation between the charges,

iii) action along the line joining the charges.

The fig.-1.1 shows two static point charged particles q_1 and at position vectors $\vec{r_1}$ and $\vec{r_j}$ respectively. Then according to The Coulomb's law, the force on j^{th} particle due to i^{th} particle will be,

$$\vec{F}_{ji} \infty \frac{q_i q_j}{r_{ij}^2} \hat{r}_{ij} \ldots \ldots \ldots (13.1)$$

Where $\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$; $r_{ji} = \left|\vec{r}_j - \vec{r}_i\right|$ and $\hat{r}_{ji} = \left(\vec{r}_j - \vec{r}_i\right) \big/ \left|\vec{r}_j - \vec{r}_i\right|$ = unit vector along $\left(\vec{r}_j - \vec{r}_i\right)$

Similarly, the force on ith particle due to jth particle will be

$$\vec{F}_{ij} - \vec{F}_{ji} \infty - \frac{q_i q_j}{r_{ij}^2} \hat{r}_{ij}$$

We can write eqn, (1.1) as, $\vec{F}_{ij} - k \dfrac{q_i q_j}{r_{ij}^2} \hat{r}_{ij}, \ldots \ldots \ldots (1.13.2)$

Where k is a constantthat, depends on the intervening space and choice of unit. In this book, we will use the SI unit for wide acceptance of this unit over the globe. In this unit k = $1/4\pi\varepsilon_0 = 9 \times 10^9$ Nm^2c^{-2}. ε_0 is known the permittivity of vacuum. The value ε_0 = 8.8542×10^{-12} $C^2N^{-1}m^{-2}$.

In SI unit the unit charge is that charge which when placed at 1m away from an identical charge in vacuum the force of interaction is 9×10^9 N. This unit charged is referred as Coulomb.

If there are N number of particles bearing charge q_1, q_2, q_3 ......at poisiton vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots\ldots\ldots\ldots$, respectively then, the total force on ith particle

$$\vec{F}_i = k\sum_{\substack{j=1 \\ j\neq i}}^{j=N} \frac{q_i q_j}{r_{ij}^{2}} \hat{r}_{ij} \ldots\ldots\ldots(1.13.3)$$

Equation (1.3) explains that, the superposition Principle is applicable for this electric interaction. This means that force on a charged particle is the vector sum of the forces due to all other charged particles.

Problem - 1

A conductor possesses 80μC of positive charge. How many electrons does it have in deficit or in excess?

Solution - 1

To get this positive charge it has to liberate electrons. As each electron has magnitude of charge 1.6×10^{-19}C, so the deficit of number electron

$$n = 80 \times 10^{-6}/1.6 \times 10^{-19} = 5 \times 10^{13}$$

Problem - 2

Four point charges each of + 10μC is placed at (3m, 0, 0), (-3m, 0, 0) and (0, -3m, 0). Find the force on a charge 10μC placed at (0, 4m, 0)

Problem - 3

(b) The Electric Field

It is obvious that electric interaction is a distant force, which means that electric interaction may migrate through space without any physical contact. Now two questions arise

i) Who is the carrier of this interaction?

ii) With what speed the interaction travels.

To resolve the first question, we introduce the conception, what is known as electric field?

This field migrates with the speed of light.

Due to the presence of charge, a quality in space is developed, which is known as electric field. In case of static charge, only electric field is developed but for dynamic charge magnetic field is also developed. The interaction between the charges takes place with this field obeying Coluomb's law without an material interaction.

A space is said to possesses electric field

The magnitude of the field is named as 'electric field intensity' $\left(\vec{E}\right)$ and is defined as :

The electric field intensity at a point is the force experienced by a unit positive charge placed at that point

1. Field Due to A Point Charge

The fig. (1.2) shows a point charge q is placed at the origin '0' of the reference frame. To calculate the field intensity atpoint p at position vector $\vec{r},$ we place a test charge dq at p. (The charge dq is so small that it does not put any distortion of field pattern of q.

Now from Coulomb's law the force on the charge dq is,

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qdq}{r^2}\hat{r} \ldots\ldots(1.3.3)$$

Therefore, the electric field intensity at p is,

$$\vec{E} = \frac{\vec{F}}{dq}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \ldots\ldots(1.3.4)$$

This shows that the field pattern of a point charge is spherically symmetric but decreasing with square of the distance from the point charge. If $\vec{E}_1, \vec{E}_2, \vec{E}_3 \ldots\ldots$

Calculation of electric potential and hence field intensity

(a) For a point charge

The fig. (1) shows a point charge q at origin 0 of referene frame. To find the potential at P at position $\vec{r}$ vector we proceed as follows. The electric field intensity at P due to + q charge at 0 $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{2}\hat{r},$ if V is the potential at P then, Then the work done in transferring a unit charge from $P\left(\vec{r}\right) to\, Q\left(\vec{r} + d\vec{r}\right),$

$dV = -\vec{E}.d\vec{r} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.d\vec{r} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{2}dr,$ so the potential at P which is the work done to carry a unit charge from infinity to the point quasi-statically,

$$V = \int_{\infty}^{r} -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\text{... ...}(13.5)$$

As V is a function of r only $E = -\partial V/\partial r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ along 0P

or, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$, which is in exact coincidence with the previous result.

(b) For a uniformly charged circular ring

The fig (1.4) shows a uniform circular ring of radius a. + q Chargedistributed

uniformly over the ring. We have to find out the potential at P at a distance x from centre 0 of the ring. λ be the charge per unit length on the ring. Consider an element charge λdl at A (in fig) Then potential at P due this element of charge,

$dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r}$, So the total potential at P du to the whole

ring $V = \oint \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\oint dl}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{a^2+x^2}}\text{... ...}(1.3.6)$

So the intensity along x axis (since V = V (x))

$$E = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\varepsilon_0}\frac{qx}{\left(a^2+x^2\right)^{3/2}} \text{ along 0P}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\vec{x}}{\left(a^2+x^2\right)^{3/2}}\text{... ... ...}\left(1.3.7\right)$$

(c) For uniformly charged disc

The fig (1.5) shows a uniformly charged disc of radius R and of charge density σCm^{-2}. To calculate the electric field intensity E, at point P at a distance × from 0 the centre of the disc we consider an elemental ring of radius r and thickness dr as in fig ( ).

The area of the elemental ring = 2π r dr

The charge of the elemental ring dq = 2πr drσ

So the potential due this elemental ring at P, $dV = \frac{1}{4\pi\varepsilon_0} \frac{2\pi r dr\sigma}{4\pi\varepsilon_0 \left(r^2 + x^2\right)^{1/2}}$

We put $r = x\tan\phi,\ dr = x\sec^2\phi d\phi,$ so, $dV = \frac{\sigma}{2\varepsilon_0}(\sin\phi)\left(x\sec^2\phi d\phi\right)$

$$V = \int_0^\theta \frac{\sigma}{2\varepsilon_0}(\sin\phi)\left(x\sec^2\phi d\phi\right) = \frac{\sigma x}{2\varepsilon_0}\left[\frac{1}{\cos\theta} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + x^2} - x\right] \text{... ...}(1.3.8)$$

As V is a function of x only, So

$$E = -\frac{\partial V}{\partial x} = -\frac{\sigma}{2\varepsilon_0}\left[-\frac{x}{\sqrt{R^2 + x^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}[1 - \cos\theta] = \frac{q}{2\pi\varepsilon_0 R^2}[1 - \cos\theta] \text{ along 0P}$$

... ... (1.3.9)

Alternative method

Field at P due this elemental ring $dE = \frac{1}{4\pi\varepsilon_0} \frac{2\pi r dr\sigma x}{\left(r^2 + x^2\right)^{3/2}}$ (ref. eqn no. (1.10))

$$= \frac{1}{4\pi\varepsilon_0 x^2} \frac{2\pi r dr\sigma x^3}{\left(r^2 + x^2\right)^{3/2}} = \frac{1}{4\pi\varepsilon_0} \frac{2\pi r dr\sigma\cos^3\theta}{x^2}$$ Now, $r = x\tan\theta,$ so, $dr = x\sec^2\theta d\theta$

So the above equation boils to,

$$dE = \frac{1}{4\pi\varepsilon_0} 2\pi\tan\theta\sec^2\theta\sigma\cos^3\theta d\theta = \frac{1}{4\pi\varepsilon_0} 2\pi\sigma\sin\theta d\theta$$

Thus the total electric field at P,

$$E = \frac{1}{4\pi\varepsilon_0} 2\pi\sigma\int_0^{\theta_1} \sin\theta d\theta = 2\pi\sigma(1 - \cos\theta) = \frac{1}{2\pi\varepsilon_0} \frac{q(1 - \cos\theta)}{a^2}$$ along 0P, same as eqn.

................(1.3.10).

(d) For a uniformly charged spherical shell :

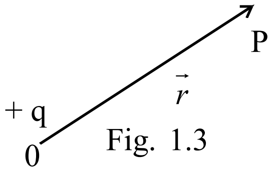


Fig. 1.3

The fig (1.6) shows a spherical shell of radius R carrying a charge q uniformly distributed over the sphere. We have to find out the potential at point P at a distane x as shown in fig (1.6) We consider an elemental ring within the angle θ and θ + dθ as in fig (1.6)

The area of the elemental ring = 2πR sin θ (Rdθ), σ be the charge per unit area.

Therefore the charge on the elemental ring $= 2\pi\sigma R\sin\theta(Rd\theta)$

The potential at P due to this ring $dV = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma R\sin\theta(Rd\theta)}{r}$

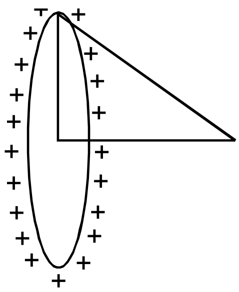
Now from the fig. (1.6) $r^2 = R^2 + x^3 - 2Rx\cos\theta$ or $2rdr = 2Rx\sin\theta d\theta$

$\Rightarrow \frac{\sin\theta d\theta}{r} = \frac{dr}{Rx}$

Therefore the above equation can be written as $dV = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma R}{x}dr$

So $V = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma R}{x}\int dr,$ where the integration is to be carried out with proper limit.

When the point P is outside sphere



$$V = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma R}{x}\int_{x-R}^{x+R} dr = \frac{1}{4\pi\varepsilon_0}\frac{4\pi\sigma R^2}{x} = \frac{1}{4\pi\varepsilon_0}\frac{q}{x} ...(1.3.11)$$

Be the intensity at a point due to a discrete distribution charges at a point, thenform principle of superposition the net field at that point will be $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 ....$

(a) Field Due To A Continuous Chage Distribution

So far we have confined to discrete charge distribution, but in macroscopic world we frequently encounter the cases where charge distribution is continuous for example a charged metallic body.

To calculate the field intensity at the point p due to such a continuous charge distribution. We usually follow this procedure. We have taken an element of charge dq as in fig (1.7) The field at p due to this elemental charge,

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$

So the net field at the point p will be,

$$\vec{E} = \int d\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\vec{r} \ldots\ldots\ldots (1.3.1)$$

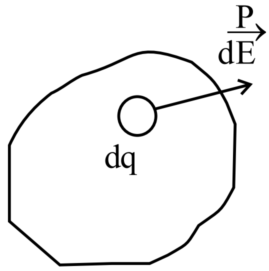


Fig. 1.1

Where the intagration is to be carried out over the entire charge distribution. Now in case of line distribution of charge dq = λ dl, where λ is the charge per unit length on the line element dl.

In case of surface distribution of charge dq = σ ds, where σ is the charge density over the surface element ds.

For a volume distribution of charge dq = ρ dv.

Electric Lines Of Force Or Field Lines

To give as visual representation of electric field the conception electric lines of force (now referred as field lines) was introduced by Michael Faraday. The electric field line is an imaginary line drawn in the space containing electric field such that tangent at any point on the line is along the direction of electric field at that point.

The field lines bear the following properties :

a) They emanate from a positive charge and end up to a negative charge or to in finity.

b) They try to contract lengthwise and repel each other laterally.

c) Field lines can't intersect each other.

d) The number of field lines associated with a charge is finite and proportional to the magnitude of charge.

e) The magnitude of field intensity at a point is the number of field lines passing per unit area when the area is placed perpendicular to the field lines.

(c) Conservative Nature Of Electric Field & Electric Potential

Now let us explore how this force field is related to energy and work. We have already seen how the electric interaction in a region can be represented by the vector $\vec{E}$, the electric field intensity. Can we represent this field in terms of a scalar called electric potential? The answer to this poser is yes.

We have already seen that electric field due to a static point charge is given by the equation

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{\left|\vec{r} - \vec{r}_0\right|^3} \left(\vec{r} - \vec{r}_0\right)$$

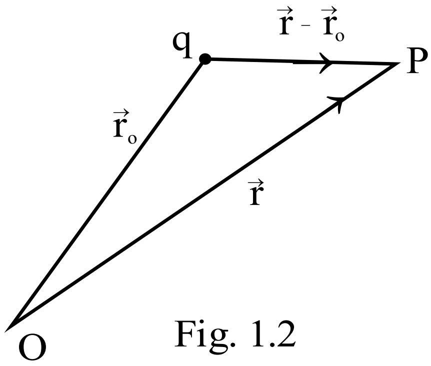


Fig. 1.2

So, $$\nabla \times \vec{E} = \frac{1}{4\pi\varepsilon_0} \nabla \times \left[\frac{q}{\left|\vec{r} - \vec{r}_0\right|^3} \left(\vec{r} - \vec{r}_0\right) \right]$$

So, $$\nabla \times \vec{E} = \frac{1}{4\pi\varepsilon_0} \nabla \times \left[\frac{q}{\left|\vec{r} - \vec{r}_0\right|^3} \left(\vec{r} - \vec{r}_0\right) \right] = \nabla \times G(R)\vec{R} = G(R)\nabla \times \vec{R} + \nabla G(R) \times \vec{R},$$

where $\vec{R} = \left(\vec{r} - r_0\right)$

$$G(R)\nabla \times \vec{R} + \nabla G(R) \times \vec{R} = G(R) \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - x_0) & (y - y_0) & (z - z_0) \end{vmatrix} + \nabla G(R) \times \vec{R}$$

So electric field is non-rotational. As curl of a gradient of a scalar is always zero i.e., $\nabla \times \vec{E} = 0 \Rightarrow \vec{\nabla} \times \vec{\nabla} V = 0$

Therefore, we can write $\vec{E} = -\nabla V \ldots (1.3.2)$

The negative sign is to carry on a logical convention that work is done in quest of electrostatic energy.

From eqn. 1.3.2 It is obvious that $\vec{E}$ remains same if V is replaced by $V + c$ *(const).* Thus, an absoulate value of potential bears indeterminacy. It depends on the choice of origin. As the electric field is zero at an infinite distance from a charge, we usually refer this point to be of zero potential. With this choice we can put c = 0. However, this constant is not so important as it does not affect the force field.

Now we take a migraction of a unit + ve charge from the space point I along the loop $i \to I \to f \to II \to i$ then the work done,

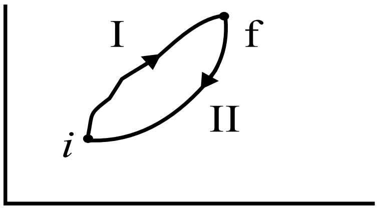


Fig. 1.3

$$W = \oint_{ilflli} \vec{E}.d\vec{r} = \oint -\nabla\phi.d\vec{r} = -\oint d\phi = 0$$

Thus $\left[\int_i^f \vec{E}.d\vec{r}\right]_{a\log ilf} + \left[\int_f^i \vec{E}.d\vec{r}\right]_{along\ flli} = 0,$

So, $\left[\int_i^f \vec{E}.d\vec{r}\right]_{along\ ilf} = \left[\int_i^f \vec{E}.d\vec{r}\right]_{along\ flli}$ that shows work done along the path i

$i \to I \to f$ and $i \to II \to f$ are same. Thus, the work done is independent of path and depends on the initial and final position. As work done is energy concerned, so in this force field, the work done depend on some energy function which solely depends on the energy of initial and final position. Such work which is a function of position is known as potential energy, here referred as electric potential or potential energy per unit charge and is represented by V.

The electric potential at a point is the work done to bring a unit positive charge from infinity up to that point quasistatically.

1.4 The electrostatic potential

It is defined as the amount of work energy needed to move a unit of electric charge from a reference point to a particular point in an electric field, precisely, it is the energy per unit charge for a test charge that is so small that the disturbance of the field under comsideration is negligible

The electric potential at a point r in a static electric field $\vec{E}$ is given by the integral

$$V_E = -\int_C \vec{E}\cdot dl \ ....(1.4.1)$$

where C is an arbitrary path from some fixed reference point to $\vec{r}$, in electrostatics, the Maxwell-Faraday equation reveals that cure $\vec{E} = 0$, making the electric field conservative. So the integral above does not depend on any speafic path c chosen, but only an end points, implying V_E is well defined at evey point. Therefor we can write $\vec{E} = -\vec{\nabla} V_E \ ...(1.4.2)$

This states that the electric field points downhill towards lower voltage. The scalar potential can be visualized using equipotential, surfaces. An equipotential surface is a surface over which is a constant. The electric field is the negative of the gradient of the electric scalar potential. The electric field lines are every where normal to the equipotential surface and point in the direction of increasing potential.

1.4.1 Electric dipole

Two equal but opposite point charges separated by a small distance consitutute an electric dipole.

The dipole moment of a dipole has magnitrde, charge time the distance between the charges and is directed from + ve to – ve charge.

If + q and – q be the charges shown. in fig (1.4). Then dipole moment of this dipole

$\vec{p} = q\left(2\vec{l}\right)$, where $\vec{l}$ is taken along $+q \overset{\leftarrow 2l \rightarrow}{\text{———}} -q$ Fig. 1.4

(a) Electric Potential due to dipote

The fig (1.5) shows an electric dipole AB. The potential at point P at position Vector $\vec{r}$ from centre O of dipole,

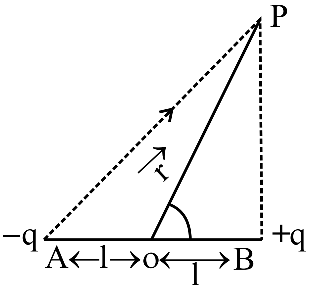


Fig. 1.5

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{BP} - \frac{q}{AP}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{r}-\vec{l}\right|} - \frac{1}{\left|\vec{r}+\vec{l}\right|}\right]$$

Taking r > > l we can write $= \dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{1}{\sqrt{r^2+l^2-2rl\cos\theta}} - \dfrac{1}{\sqrt{r^2+l^2+2rl\cos\theta}}\right]$

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2-2rl\cos\theta}} - \frac{1}{\sqrt{r^2+2rl\cos\theta}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\sqrt{1-\frac{2l\cos\theta}{r}}} - \frac{1}{\sqrt{1+\frac{2l\cos\theta}{r}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\left(1+\frac{l\cos\theta}{r}\right)-\left(1-\frac{l\cos\theta}{r}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{2l\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta}{r^2}$$

or, $V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} \ldots.(1.4.3)$

So the electric field intensity at P

$$\vec{E} = -\vec{\nabla}V = -\vec{\nabla}\frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{P}\cdot\vec{r}}{r^3}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^3}\vec{\nabla}\left(\vec{P}\cdot\vec{r}\right)+\left(\vec{P}\cdot\vec{r}\right)\vec{\nabla}\frac{1}{r^3}\right] \ldots.(1.4.4)$$

Calculation

$$\nabla\left(\vec{P}\cdot\vec{r}\right) = \vec{\nabla}\left(\hat{i}P_x+\hat{j}P_y+\hat{k}P_z\right)\cdot\left(\hat{i}x+\hat{j}y+\hat{k}z\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(xP_x+yP_y+zP_z\right)$$

$$= \hat{i}P_x+\hat{j}P_y+\hat{k}P_z = \vec{P},$$

and $\vec{\nabla}\left(\frac{1}{r^3}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left[\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\right]$

$$= \hat{i}\left\{-\frac{3}{2}\frac{2x}{\left(x^2+y^2+z^2\right)^{5/2}}\right\}+\hat{j}\left\{-\frac{3}{2}\frac{2y}{\left(x^2+y^2+z^2\right)^{5/2}}\right\}+\hat{k}\left\{-\frac{3}{2}\frac{2Z}{\left(x^2+y^2+z^2\right)^{5/2}}\right\}$$

$$= 3\frac{\left(\hat{i}x + \hat{j}y + \hat{k}z\right)}{r^5} = -3\frac{\vec{r}}{r^5}$$

$$\text{So } \vec{E} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{P}}{r^3} - \frac{3\left(\vec{P}\cdot\vec{r}\right)\vec{r}}{r^5}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{3\left(\vec{P}\cdot\vec{r}\right)\vec{r}}{r^5} - \frac{\vec{P}}{r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[\frac{3\left(\vec{P}\cdot\vec{r}\right)}{r^2}\vec{r} - \vec{P}\right] \text{....}(1.4.5)$$

So the radial component in (r,θ) co-ordinate

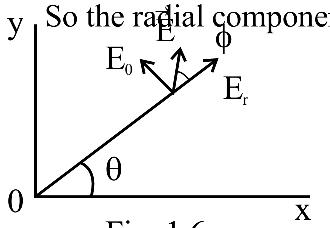


Fig. 1.6

$$E_r = \frac{1}{4\pi\varepsilon_0 r^3}\left[3\frac{\mathrm{Pr}^2\cos\theta}{r^2} - P\cos\theta\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2P\cos\theta}{r^3}\text{....}(1.4.6)$$

and the transverse component

$$E_\theta = \frac{1}{4\pi\varepsilon_0 r^3}\left[O - \left(-P\sin\theta\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}P\frac{\sin\theta}{r^3}\text{....}(1.4.7)$$

So $\vec{E}$ makes an angle ϕ with E_r

$$\tan\phi = E_\theta / E_r = \frac{1}{2}\tan\theta\text{....}(1.4.8)$$

Thus along with same line when r > > l field lines are parallel to each other on same $\vec{r}$.

$$\left|\vec{E}\right| = \sqrt{E_r^2 + E_\theta^2} = \frac{1}{4\pi\varepsilon_0}\frac{P}{r^3}\left(3\cos^2\theta + 1\right) \ldots\ldots(1.4.9)$$

field at end-on position (on the axis of dipole) along $\vec{p}$

$$\vec{E}_{end-on} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{P}}{r^3} \ldots\ldots(1.4.10)$$

field at broad side position (on perpendicular bisector)

$$\vec{E}_\theta = \frac{-\vec{P}}{4\pi\varepsilon_0 r^3} \ldots\ldots(1.4.11)$$

(b) Dipole in a electric field.

P.E of a dipole in electric field.

The potential energy (P.E) of a dipole in a electric field is the work done to bring the dipole from infinity to the point concerned quasislatically Now if a dipole is brought to a point with its poles perpendicular to the direction of field then the work done will be zero since + ve and – ve charge will do same work but in opposite direction. So we can take the field normal position to be zero P.E Position.

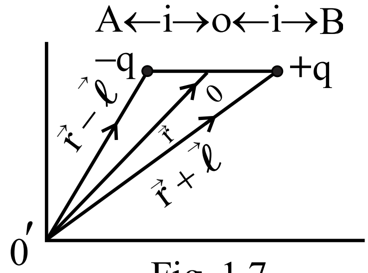


Fig. 1.7

The fig. (1.7) shows a dipole AB of length $2l.V\left(\vec{r}\right)$ P.E the potential energy at 0, the centre of dipole. Then the P.E of the dipole.

$$U = -qV\left(\vec{r} - \vec{l}\right) + qV\left(\vec{r} + \vec{l}\right)$$

Taking $\vec{l}$ to be very very small

$$U = -q\left\{V\left(\vec{r}\right) - \vec{l}\cdot\vec{\nabla}V\left(r\right)\right\} + q\left\{V\left(\vec{r}\right) + \vec{l}\cdot\vec{\nabla}V\left(\vec{r}\right)\right\}$$

$$= 2q\vec{l}\cdot\vec{\nabla}V\left(\vec{r}\right) = \vec{P}\cdot\vec{\nabla}V\left(\vec{r}\right) = -\vec{P}\cdot\vec{E}$$

$$\left(\text{As}\,\vec{E} = \vec{\nabla}V\left(\vec{r}\right)\right)$$

So the force on the dipole

$$\vec{F} = -\nabla U = \vec{\nabla}\left(\vec{P}\cdot\vec{E}\right)$$

Using the vector identity

$$\vec{\nabla}\left(\vec{P}\cdot\vec{E}\right)=\left(\vec{E}\cdot\vec{\nabla}\right)\vec{P}+\left(\vec{P}\cdot\vec{\nabla}\right)\vec{E}+\vec{E}\times\left(\vec{\nabla}\times\vec{P}\right)+\vec{P}\times\left(\vec{\nabla}\times\vec{E}\right)$$

We have

$\vec{F}=\left(\vec{P}\cdot\nabla\right)\vec{E}$ ....(1.4.18) since the rest of all terms are individually zero.

(c) Torque on a dipole in uniform electric field.

The fig (1.8) shows a dipole AB in uniform electric field $\vec{E}.$ The total fore on the dipole is zero since poles are acted by equal and opposite forces. The torque on the dipole $\tau = qE\,21\sin\theta = PE\sin\theta$

Taking direction of $\vec{\tau}=\vec{P}\times\vec{E}$.....(1.4.13)

If the field is non-uniform apart from the torque there will be a translational force also on dipole.

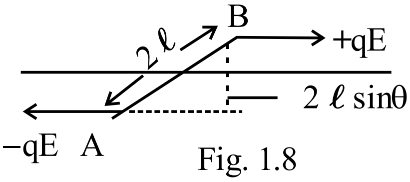


Fig. 1.8

(d) Dipole–dipole interaction

Now we shall calculate the potential energy of due to dipole–dipole interaction of two co-planar dipoles.

The fig (1.9) shows two coplanar dipole of dipole moments $\vec{P}_1$ and $\vec{P}_2$ at position vector $\vec{r}_1$ and $\vec{r}_2$ respectively. $\vec{r}=\left(\vec{r}_2-\vec{r}_1\right)$.

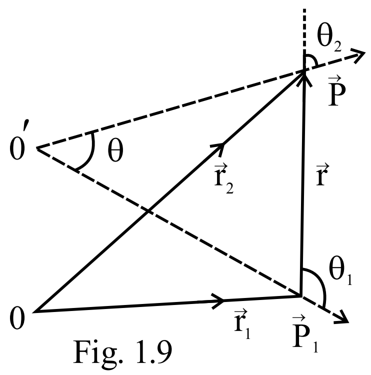


Fig. 1.9

The potential energy of dipole of dipole moment $\vec{P}_2$ in the field of dipole of dipole moment $\vec{P}_1$,

$U_{21}=-\vec{P}_1\cdot\vec{E}_1$, where $\vec{E}_1$ is the field due to $\vec{P}_1$ at $\vec{P}_2$.

$$=-\vec{P}_2\cdot\frac{1}{4\pi\varepsilon_0 r^3}\left[\frac{3\left(\vec{P}_1\cdot\vec{r}\ \right)\vec{r}}{r^2}-\vec{P}_1\right]$$

$$=-\frac{1}{4\pi\varepsilon_0 r^3}\left[3\frac{\left(\vec{P}_1\cdot\vec{r}\right)\vec{P}_2\cdot\vec{r}_1-\left(\vec{P}_2\cdot\vec{P}_1\right)}{r^2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[P_1P_2\cos\theta - 3P_1\cos\theta_1 P_2\cos\theta_2\right]$$

$$= \frac{P_1P_2}{4\pi\varepsilon_0 r^3}\left[\cos(\theta_2-\theta_1) - 3\cos\theta_1\cos\theta_2\right]$$

$$= U_{12}$$

Now if the dipoles have same dipole moment i.e. $\vec{P} = \vec{P}_2$ i.e they are oriented along same line,

Then $U = \dfrac{P^2}{4\pi\varepsilon_0 r^3}[1-3] = \dfrac{-P^2}{2\pi\varepsilon_0 r^3}$.

1.5 Multiple Expansion of Electrostatic Potential

It has been carlier seen that when the charge distribution has sufficient symnetry, or when the potential is to be calculated on symnetry axis of the distribution, one can obtain potential due to it a point outside the charge distribution by solving Laplace's equation with necessary boundary condition. It is a problem of finding potential due to an arbitrary charge distribution and one must take recourse to appronimation. One such proceedure, called multiple expansion of the potential, at a point far removed from the distribution. In this appronimation, the potential is expressed as a sum of contribution due to charge monopole, dipole, and quadrapole etc

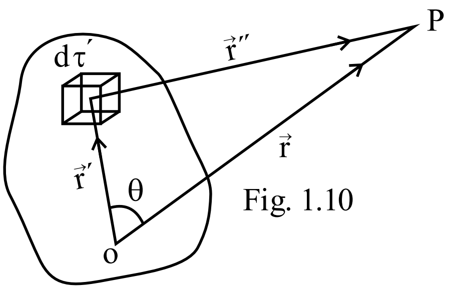


Fig. 1.10

Let us consider a charge distribution shown in the figure (1.10). The potential at the point $\vec{r}$ is given by

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r''}\rho(r')\,d\tau' \quad ....(1.5.1)$$

Using the law of cosines,

$$r''^2 = r^2 + (r')^2 - 2rr'\cos\theta$$

$$= r^2\left[\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

where θ is the angle between $\vec{r}$ and $\vec{r}'$ so we get, $r'' = r\sqrt{(1+\varepsilon)}$

with $\varepsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$

For points far from the charge distribution ε < 1, from binomial expansion.

$$\frac{1}{r''} = \frac{1}{r}(1+\varepsilon)^{1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + ...\right)$$

or interms of r, r' and θ

$$\frac{1}{r''} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta\right)^3 + ...\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \right]$$

In the last expansion, we sbserve that series comes with power of $\frac{r'}{r}$ along with Legerdre polynomial as coefficients. we get

$$\frac{1}{r''} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta) ....(1.5.3)$$

Substituting the equatim 1.5.3 in equation (1.5.1) we get the potential as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\rho(\vec{r}')d\tau'$$

More explicitly,

$$V\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho\left(r'\right)d\tau'+\frac{1}{r^2}\int r'\cos\theta\,\rho\left(r'\right)d\tau'+\frac{1}{r^3}\int\left(r'\right)^2\right.$$

$$\left.\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho\left(r'\right)d\tau'+....\right]....(1.5.5)$$

The equation (1.5.5) is expression for multipole expansion of V in powers of $\frac{1}{r}$.

rearrange the term as follows–

$$V\left(\vec{r}\right)=V_0\left(\vec{r}\right)+V_1\left(\vec{r}\right)+V_2\left(\vec{r}\right)+.....(1.5.6)$$

where, $V_0\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon r}\int\rho' d\tau'.....(1.5.7)$

$$V_1\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\theta\rho\left(r'\right)d\tau'....(1.5.8)$$

$$V_2\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0 r^3}\int r'^2\frac{1}{2}\left(3\cos^2\theta-1\right)\rho'\left(r'\right)d\tau....(1.5.9)$$

The Monopole and Dipole Terms :

Monopole term is defined as.

$$V_0\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0 r}\int\rho'\left(r'\right)d\tau'.....(1.5.10)$$

It is the potential which would have at P if The whole charge is concentrated at the origin, and $\int\rho d\tau'$ is the monopole moment.

If the total charge is zero, the dominant term in the potential will be the dipole.

$$V_{dip}\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\theta\rho\left(r'\right)d\tau'....(1.5.11)$$

Since θ is the angle between $\vec{r}'$ and $\vec{r}$

$r' \cos\theta = \hat{r} \cdot \vec{r}'$

So we can write the dipole term as

$$V_{dip}\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}.\hat{r}\int r'\rho\left(r'\right)d\tau' \ldots.(1.5.12)$$

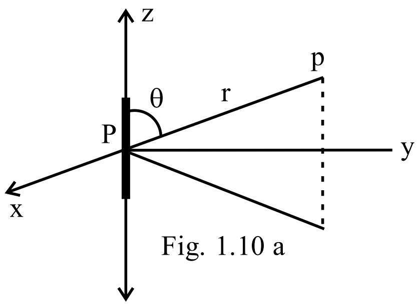


Fig. 1.10 a

This integral is called the dipole moment of the charge distribution,

$$\vec{P} = \int \vec{r}'\rho\left(r'\right)d\tau' \ldots.(1.5.13)$$

So the dipole contribution to the potential is given by

$$V_{dip}\left(r\right) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}\cdot\hat{r}}{r^2} \ldots.(1.5.14)$$

The dipole term plays the important role when all the monopole term vanishes, so $V_1\left(\vec{r}\right)$ is the potential as if a pure dipole is placed at the origin. The term $V_2\left(\vec{r}\right)$ is defined as potential contribution due quadrapole moment The figure 1.10a portrays gemetrical

1.6 The Gauss's Theorem

(a) Electric flux :

In science, flux, usually concerns to some flow of physical property. In case of fluid flow 'fluid flux' refers to the amount of fluid flowing through a specific area per unit time. 'Vehicular flux' often refers to the number vehicle crossing a specific gate area per unit time. However, in case of electric flux, no such transport physically exists. It refers to the crossing of electric field lines (which is an imaginary conception to give a visual presentation of field pattern) through specific area. It is defined as :

The electric flux through an area the number of field lines passing perpendicular to the area. $\vec{E}$ be the electric field at a space point. Then the flux through a surface, $d\vec{s}$ is,

$$d\Phi = \vec{E}.\vec{ds} \ldots.$$

As $d\vec{s}$ is infinitesimal $\vec{E}$ can be taken to be constant over the surface. The total flux on a finite surface can be obtained by integration as,

$\Phi = \int \vec{E}.d\vec{s},$ where the integration is carried over the entire surface.

(b) Surface area and solid angle :

Surface area is treated as a vector whose magnitude is the area of the surface considered and direction is specified as follows.

i) For closed surface the direction is + ve in outwards normal to the surface.

ii) In case of open surface the direction is specified by right hand screw rule.

The **solid angle** is three dimensional analogues to that of an **angle** in two dimension.

Now an angle can be visualized physically as a two dimensional peeping from a point. Mathematically it is defined as the ratio of the arc by radius of a circle.

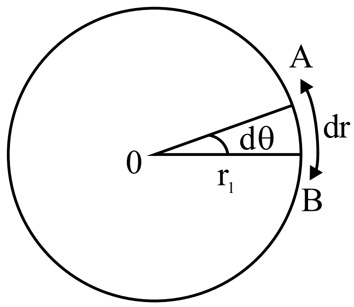


Fig. 1.11

In fig the angle subtended

$d\theta = \text{arc AB}/r = dl/r,$ which is a dimensionless quantity. Its **unit** is taken as radian which is defined as angle subtended by a arc of a circle of unit length and unit radius at the centre of the circle, so, the total angle about a point as $2\pi\frac{r^2}{r^2} = 2\pi$ radian.

Similarly, a solid angle can be visualized as a three dimessional peeping through a point and it is mathematically defined as three dimensional angle produced at the centre of a sphere due to an area boudary on the surface of the sphere.

If *ds* be the elemental area in the surface of a sphere then, the solid angle suspended at the centre O is, $d\Delta\Omega = \frac{ds}{r^2}$

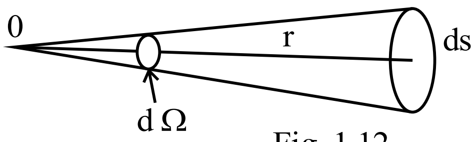


Fig. 1.12

If the area plane makes an angle θ with the tangent to the sphere at that point then solid angle $d\Omega = \frac{ds\cos\theta}{r^2} = \frac{d\vec{s}.\hat{r}}{r^2} = \frac{ds\vec{n}.\hat{r}}{r^2}$. where $\hat{n}$ and $\hat{r}$ are unit vectors along d $\vec{s}$ and $\vec{r}.$ as explained in the figure.

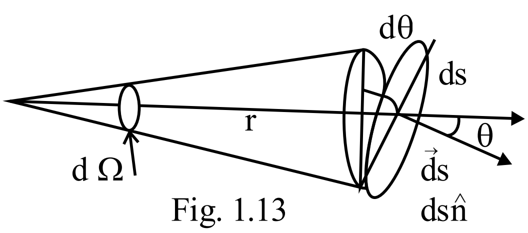


Fig. 1.13

Unit of solid angle is named steradian, one steradian is the solid angle subtended at the centre of a sphere of unit radius by a unit surface on the sphere.

Obviously the total solid angle about a point will

$$\Omega = \oint \frac{ds\hat{n}.\hat{r}}{r^2} = \oint \frac{ds\cos\theta}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradian}$$

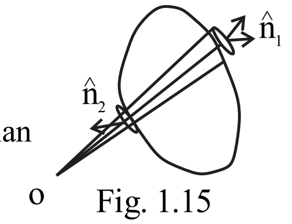


Fig. 1.15

When o is outside then referring the fig.

Due to the orientation of $\hat{n}_1$ *and* $\hat{n}_2$ the surface area vectors the total solid angle subtended at o will obviously be zero, because they will cast solid angle of same magnitude but in opposite sense and will cancel each other to make the yield null.

(c) The Gauss's law :

Now we consider a point charge q at a point o bounded by the surface S. The electric field on the surface ds is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$, so the flux through the surface element ds,

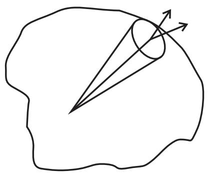


Fig. 1.14

$$d\Phi = \vec{E}.d\vec{s} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.d\,s\hat{n} = \frac{1}{4\pi\varepsilon_0}qd\Omega$$

So the total flux outgoing the whole surface S,

$$\Phi = \oint_S \frac{1}{4\pi\varepsilon_0}qd\Omega = \frac{1}{4\pi\varepsilon_0}q.4\pi = \frac{q}{\varepsilon_0} \text{......}(1.6.1)$$

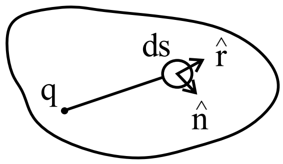


Fig. 1.11

The result in equation is independent of position of the charge and obviously when the point charge q falls outside the surface the yield integration is zero, in that case

$$\Phi = 0$$

1.6 Application of Gauss's theorem

Before going to the application of Gauss's theorem let us give a second look to what we have done in the previous discussion. We see that with Coulomb's law, if we know the charge, we are able to find out the field produced by the charge and from Gauss's law if the field in a region is known, we can work out the net charge responsible for creating the field if we can evaluate $\int \vec{E}, d\vec{s} = \int Eds \cos\theta$ which is obviously not a function of single variable. So for evaluation of this integral the angle between $d\vec{s}\, and\, \vec{E}$ should remain constant throughout the surface. Such a hypothetical surface, on which such symmetry is maintained, so that $\int \vec{E}, d\vec{s}$ can be evaluated, is called **Gaussian Surface.**

(1) Field due to a uniformly charged spherical shell :

The fig (1.17) shows a hollow spherical charged sphere of radius R and charge q, uniformly distributed over its surface. To calculate the intensity at point P at a distance r form the centre o we consider a Gaussian surface shown by dotted line in the fig (1.17). E be the intensity of field at a distance r from centre 0. Then using Gauss's law.

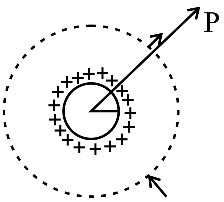


Fig. 1.12

$$\int \vec{E}, d\vec{s} = \int Eds \cos\theta = E4\pi r^2 = q/\varepsilon_0 \text{ or, } E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \ldots\ldots(1.6.2)$$

Inside the shell the right hand side is zero as there is no charge included within the Gaussian surface for any value of r (r < R) so E = 0 within the shell.

(2) Field due to a uniformly charged sphere :

(a) At a point outside the surface :

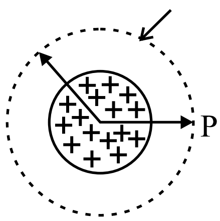


Fig. 1.13

The fig (1.18) shows a uniformly charged sphere of radius R and total charge q uniformly distributed. To calculate the field intensity E at the point P outside the surface, we consider a Gaussian surface through the point P as shown by the dotted line. $\vec{E}$ be the field intensity at point P and $d\vec{s}$ be an elemental area on the surface at point P, then by Gauss's theorem

$$\oint \vec{E}, d\vec{s} = \frac{q}{\varepsilon_0} \ldots\ldots(1.6.3)$$

Now from symmetry of charge and its consequent field distribution E remains same all over the Gaussian surface and is always on the surface. Hence,

$$E\left(4\pi r^2\right) = \frac{q}{\varepsilon_0}, \text{ or } E = \frac{q}{4\pi\varepsilon_0 r^2}$$

Taking direction into consideration $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}......(1.6.4)$

($\hat{r}$ is a unit vector in the direction of $\vec{r}$.)

(b) At a point inside the surface :

To calculate the field inside the surface, we consider the Gaussian surface represented by dotted line as shown in fig (1.14) R be the radius of the sphere and ρ be the charge density taken to be uniform inside the sphere. E be the intensity over the Gaussian surface. Then,

$$\oint \vec{E}.d\vec{s} = \left(\frac{4}{3}\pi r^3\rho\right)\Big/\varepsilon_0 \text{ or } E\left(4\pi r^2\right) = \left(\frac{4}{3}\pi r^2\rho\right)\frac{R^3}{R^3}\Big/\varepsilon_0\Big)$$

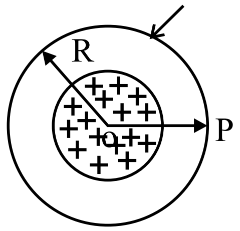


Fig. 1.14

Thus $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\vec{r}......(1.6.5)$

From the above equation it is obvious that electric field of a uniformly charged sphere is zero at the centre of the sphere and linearly increase up to the surface of sphere where it resumes its maximum value. For outside the surface the field falls $1/(\text{distance})\frac{1}{(\text{distance})^2}$ from the centre of the sphere.

(3) A uniformly charged long cylinder :

Consider an infinitely long cylinder having uniform linear charge density and radius a. Let P be a point located at a perpendicular distance or from the wire we construct the Gaussian surfaces, wehich in this case is a concetric cylinder of radius r and length ℓ.

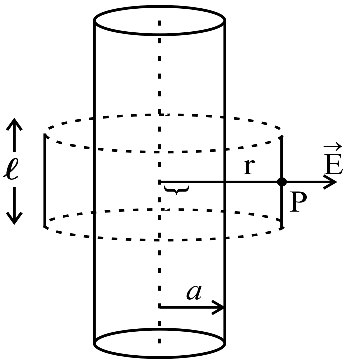


Fig. 1.15

Applying Gauss's law for this surface, we get $\int_S \vec{E}.d\vec{s} = \frac{1}{\varepsilon_0}\lambda\ell$

Where $\lambda\ell$ is the charge enclosed by the surface. Electrical field lines $\vec{E}$ will be normal to the curved portion of the surface. Due to cylindrical symmeiry, $\vec{E}$ will be of same magnitude all over it. Also $\vec{E}$ will be tangential to the end faces, so $\vec{E}.d\vec{s}$ will be zero on these faces, we can write

$$\int_S \vec{E}.d\vec{s} = E\int ds = E.2\pi rl = \frac{1}{\varepsilon_0}\lambda\ell.$$

or, $E = \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{1}{r} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda}{r}$

vector notation $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda}{r}\hat{r}...(1.6.6)$

(b) Field inside cylinder :

To find the electric field at any internal point, P at a distance r, we construct a cylinder cal Gaussian surface, of length l and radius r coaxial with it. The charge endosed by the surface is $Q_{encl} = \pi r^2 l.s.$

where ρ is the charge density o to and is related to λ by $\pi a^2.\rho.1 = \lambda$, $a\rho = \dfrac{\lambda}{\pi a^2}$

From Gauss's law

$$\int \vec{E}.d\vec{s} = \frac{1}{\varepsilon_0}Q_{encl}$$

Now $\int \vec{E}.d\vec{s} = E.2\pi rl$

or, $E2\pi rl = \dfrac{1}{\varepsilon_0}Q_{encl} = \dfrac{1}{\varepsilon_0}\pi r^2 l\rho$

$= \dfrac{1}{\varepsilon_0}\pi r^2 l\dfrac{\lambda}{\pi}\dfrac{}{a^2}$

or, $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda r}{a^2}$

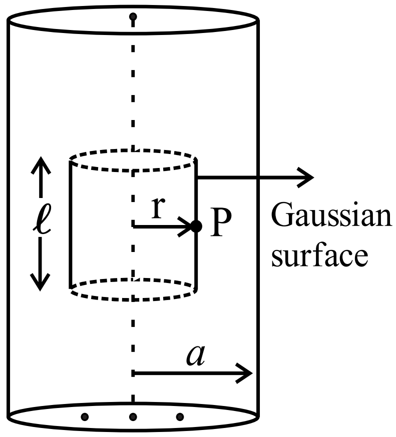


Fig. 1.16

In vector form, $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2\lambda r}{a^2}\hat{r}$ ... ...(1.6.7)

(4) Unformly charged infinite plane :

Consider an infinite plane sheet of charges with uniform surface charge density σ. To find out electric field at P at distance r, we construct a cylindrical Gaussian havig equal length on both sides

From Gauss's law

$$\Phi_E = \oint_S \vec{E}.d\vec{s} = \frac{1}{\varepsilon_0} Q_{encl}$$

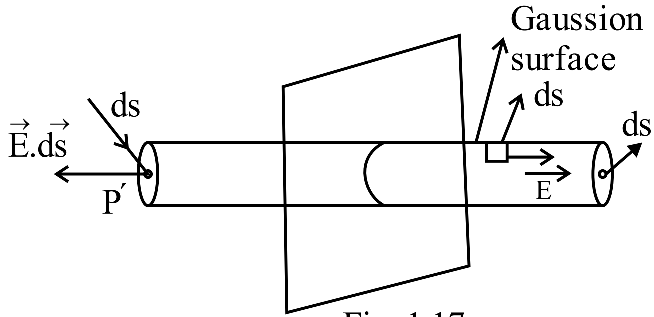


Fig. 1.17

or $$\phi = \oint \vec{E}\cdot d\vec{s} = \int_{\text{curved surface}} \vec{E}\cdot\vec{ds} + \int_P \vec{E}\cdot\vec{ds} + \int_{P'} \vec{E}.\vec{ds} = \frac{Q_{encl}}{\varepsilon_o}$$

The electric field is perpendicular to the area element at all points on the carved surface and is parallel to the surface P and P'

$$\phi_E = \int_P Eds + \int_{P'} Eds = \frac{\phi_{encl}}{\varepsilon o} \quad \left[\int_{\text{curved}} E\cdot ds = 0\right]$$

Since the magnitude of the electric fild at these two equal surfaces is uniform, E is taken out of the intergration and $Q_{encl} = \sigma , s$ $\quad 2E\int ds = \frac{\sigma s}{\varepsilon_0}$

Hence $2ES = \frac{\sigma S}{\varepsilon_0}$

or, $E = \frac{\sigma}{2\varepsilon_0}$

If $\hat{r}$ is unitvect perpendicular to the plane, then in vector form

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} \quad \ldots\ldots(1.6.8)$$

Electrical Field Inside A Parrallel Plate Conductor :

At the points P_2 and P_3, he electric field due to both plates are equal in magnitude and opposite in direction. As a result, electric field at a point outside the plates is zero. But inside, electric fields are in the same direction ie. towards the right, the total electric field at a point P_1

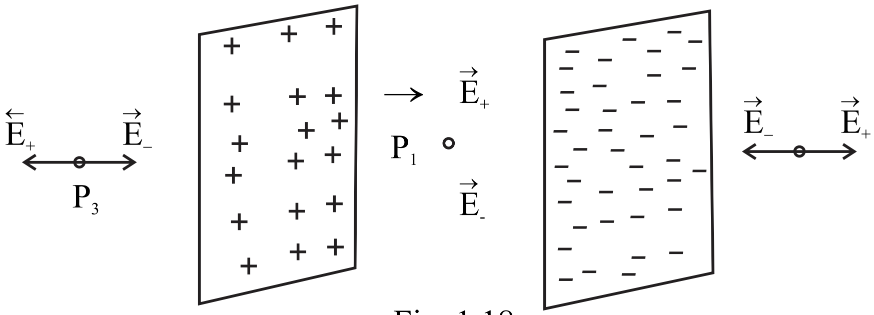


Fig. 1.18

$$E_{inside} = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \;......(1.6.9)$$

The direction of the electric field inside the plates is directed from positively charged plate to negatively charged plate and is uniform every where inside the plate

1.7 Laplace and Pisson's Equations

Consider a closed surface S enclosing a volume V and charge q. Then from Gauss's law we can write

$$\oint \vec{E}\vec{ds} = \frac{q}{\varepsilon_0} = \int_V \frac{\rho dV}{\varepsilon_0}$$

Where $\vec{E}$ electric field intensity vector at the point $d\vec{s}$ and the ρ is the charge density at the point concerned to $\vec{E}$. Using Gauss's divergence theorem in above equation we have

$$\int_V \vec{\nabla}.\vec{E}dV = \int_V \frac{\rho dV}{\varepsilon_0}$$

or, $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \;......(1.7.1)$

The equation (1.17) is known as differential form of Gauss's law of electrostatics. Now using potential – intensity relation, we have

$$\vec{\nabla}.\left(-\vec{\nabla}.\Phi\right) = \frac{\rho}{\varepsilon_0}$$

or, $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$ ... ...(1.7.2)

In free space ρ = 0 the above equation takes the form

$\nabla^2\Phi = 0$ ... ...

Equation (1.7.1) and (1.7.2) are respectively known as Poission and Laplace's equations, which play important role in solving out the potential of a charge distribution with given boundary condition, which is the basic motto of solving electrostatic problems.

∇^2 in different co-ordinate system :

(a) Rectangular co-ordinates (x, y, z)

$$\nabla^2\Phi = \frac{\partial^2\phi}{\partial n^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

(b) Spherical polar co-ordinates (r, θ, ϕ)

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

(c) Cylindrical co-ordinates (r, θ, z)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{s\Phi}{\partial s}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

1.7.1 Uniqueness Theorem

Two solutions of Laplace's equation obeying the same boundary conditions diffon at best by a constant.

In order to prove the The orem, let us assame that ϕ_1 and ϕ_2 are the two solutions of Lapalac's equation in volume V exterior to surface of different conductor S_1, S_2....S_n. bouded by on the outside surfaces. Assuming that ϕ_1 and ϕ_2 satisfy the same boundary condutions including the surfaces, S_1...S_2....S_n, and specifically. These boundary conditions includes either the specificatons of the potential ϕon the bounding surface which is known Dirichlet condition, or in other way, the specification of the normal derivati overs of ϕ i.e. $\frac{\partial\phi}{\partial n}$. The bounding surface, known as Neuman condition.

Let $\phi = \phi_1 - \phi_2$, As $\nabla^2\phi_1 = -\rho/\varepsilon_0$ and $\nabla^2\phi_2 = -\rho/\varepsilon_0$

so, $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ inside V and $\phi = 0$, an $\frac{\partial\phi}{\partial n} = 0$ on the surface S for Dirichlet and Neuman boundary conditions, respectively. Applying divergence Therem to the vector $\phi\vec{\nabla}\phi$, we get

$$\int_S \phi\vec{\nabla}\phi\cdot\hat{n}ds = \int_V \vec{\nabla}\cdot\left(\phi\vec{\nabla}\phi\right)dv$$

$$\int_V \left[|\nabla\phi|^2 + \phi\nabla^2\phi\nabla^2\phi\right]dv\backslash$$

$$\int_V |\nabla\phi|^2\, dv \quad \left[\therefore \nabla^2\phi = 0\right]$$

The integration on the left hand side vanishes on both types boudary conditions so, we get

$$\int_V \left|\vec{\nabla}\phi\right|^2 dv = 0$$

It is clear that integrand is positive definite quantity, it must be zero at every point in V for the integral to vanish, where,

$\vec{\nabla}\phi = 0$, or, $\phi_1 - \phi_2 =$ costant inside V

Now, for Diricihet boundary conditions $\phi_1 - \phi_2 = 0$ on the surface S, so we have $\phi_1 - \phi_2$ through out i.e. it is a unique solution.

For Neuman's boundary conditions

$\vec{\nabla}\left(\phi_1 - \phi_2\right)\cdot\hat{n} = 0$ on. S

or, $\phi - \phi_2 =$ Constant S

As the constant is arbitrary, it may can be taken to be zero, and the solution is unique : $\phi_1 - \phi_2$

1.8 Electrostatic Energy

The energy stored in accumulation of charge due to work done against the Coulomb'sforce, is called electrostatic energy. This is essentially potential energy in case of static charges. reference zero of potential energy can be set to be zero at that separation.

Calculation of electric potential energy for a charge distribution

Consider a point charge q_i at position vector N_i and all other charges, infinitely separated from each other. Then to bring a point charge q_j at position vector $\vec{r}_j$ the work done

$$w_{ij} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r_j} \frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij}\cdot d\vec{r}_{ij} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r_j} \frac{q_i q_j}{r_{ij}^2}dr_{ij}, \text{ where } \vec{r}_{ij} = \vec{r}_j - \vec{r}_i \text{ and } \hat{r}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{\left|\vec{r}_j - \vec{r}_i\right|}$$

$$w_{ij} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}\Bigg|_{\infty}^{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}}$$

So the total work done for accumulating N charges

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{j=1}^{j=N}\Phi_j q_j \quad ..1.8.1$$

Where $\Phi_j = \sum_{i=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_{ij}}$ is the potential due to all particles at j[th] charge point. The multiplication of 1/2 is to avoid double counting like $\Phi_1 q_2$ and $\Phi_2 q_1$ which will have same effect on energy.

Electrostatic energy for continuous charge distribution

we can replace i) $q_j = \rho(r)dV$ for volume distribution of charge,

ii) $q_j = \sigma(r)ds$ for surface distribution of charge,

iii) $q_j = \lambda(r)dl$ for line distribution of charge,

Considering all type of distribution of charge the expression of electrostatic energy

$U = \frac{1}{2}\left[\int_V \rho(r)\Phi(r)dV + \int_S \sigma(r)\Phi(r)dS + \int_L \lambda(r)\Phi(r)dl\right] +$ *the energy due to point charge e distribution.*

1) Electostatic energy in terms of field vectors

We start with a volume distribution of charge, then the electrostatic energy

$$U = \frac{1}{2}\int_V \rho(r)\Phi(r)dV = \frac{1}{2}\int_V \Phi(r)\nabla.\vec{D}\,dV$$

Now using $\nabla.\left(\Phi\vec{D}\right) = \nabla\Phi.\vec{D} + \Phi\left(\nabla.\vec{D}\right)$ In the above equation we have

$$U = \frac{1}{2}\int_V\left[\nabla.\left(\Phi\vec{D}\right) - \Phi\nabla.\vec{D}\right]dV = \frac{1}{2}\left[\int_S\left(\Phi\vec{D}\right).d\vec{S} + \int_V \vec{E}.\vec{D}\right]dV]$$

Now confining the charge to a finite region if the integration is extended to infinity the first term vanishes since $\Phi \infty\, {}^{1}\!/_{r}, D \infty\, {}^{1}\!/_{r^2}$ and $ds \infty^2$ r² so the first term falls as ${}^{1}\!/_{r}$. Thus when we extend for all space the expression of electrostatic energy becomes.

$$U = \frac{1}{2}\int_V \vec{E}.\vec{D}\,dV ...(1.8.2)$$

We can obviously take electrostatic energy density $u = \frac{1}{2}\left(\vec{E}.\vec{D}\right)$

1.8.1 Electrostatic Energy of Uniformly charged sphere :

Eectrostatic self energy of a charged sphere us given by–

$$U = \frac{1}{2}\varepsilon_0 \int_{inside} E^2 dV + \frac{1}{2}\varepsilon_0 \int E^2 dV.$$

Assuming total charge of the sphere Q, ρ its charge density and 'a' is the radius, then, electric field at a dirtaue 'r'

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q_r}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{4/3\pi r^3\rho}{r^2}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{Q\cdot r}{a^3} \text{ where } r < a.$$

and $E=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,for\ r>a$

$$\therefore U=\frac{\varepsilon_0}{2}\int_0^a\left(\frac{Qr}{4\pi\varepsilon_0 a^3}\right)^2 4\pi r^2 dr+\frac{\varepsilon_0}{2}\int_a^\infty\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^4 4\pi r^2 dr$$

$$=\frac{Q^2}{2}\cdot\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{5a}+\frac{1}{a}\right)=\frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q^2}{5a}$$

Total energy of charged sphere,

$$U=\frac{1}{4\pi\varepsilon_0}\frac{3Q^2}{5a} \ldots\ldots(1.8.3)$$

1.9 Conductors in electric field

When a conductor is placed in electric field the free electrons in it move in opposite direction creating induced field in opposite direction as in fig. The electron migration continous untill the field inside the conductor vanishes and the conductor becomes equi-potential all through. Any charge given to the conductor will migrate to its surface. If not

$\nabla.E=\frac{\rho}{\varepsilon}$ implies presence of electric field inside the conductor. Which contradicts the above discussion.

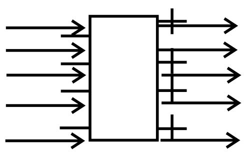


Fig. 1.19

1) Field intensity on the surface of a charged conductor.

Consider a cylinder a cylinder of plane surface area $d\vec{s}$ as in fig 1.2.9 The electric field at any point of the surface is perpendicular to the surface as $\vec{\nabla}\Phi$ is perpendicular to constant potential surface. Now consider a Gaussian surface as denoted by a cylinder of each plane surface $d\vec{s}\cdot\vec{E}$ be the intensity at the surface point at the element $d\vec{s}$. The using Gauss's law we have $\vec{E}.d\vec{s}=\frac{\sigma ds}{\varepsilon}$ or, $\vec{E}=\frac{\sigma}{\varepsilon}$ acting normally outwards at the point considerd ... ... (1.9.3)

2) Mechanical Force and presure on a charged conducting surface

Consider a charged coductor as in fig. (1.20) then by Coulomb's Theorem the electric field at vicinity of the outside of the surface is $E = \sigma/\varepsilon$ acting normally outwards. We can visualize the $\vec{E} = \vec{E}_1 + \vec{E}_2$; where $\vec{E}_1$ is the field intensity due to the charge on the element ds and $\vec{E}_2$ is the field due to the rest of the charge at the element ds. Now the intensity inside the conductor is zero, so $\vec{E}_1 + \vec{E}_2 = 0$, thus $E_1 = E_2 = \sigma/2\varepsilon_0$ and at outside point the intensity will match with the Coluomb's theorem

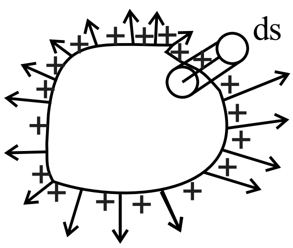


Fig. 1.20

$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sigma/\varepsilon_0$ acting outwards normal.

Thus the elemental charged area σds will be under the outwards field intensity $E_1 = \sigma/\varepsilon_0$

So the force on the charge elemental

$$\Delta\vec{F} = \frac{\sigma^2 ds}{2\varepsilon_0}\hat{n}\left(\hat{n} = unit\ vector\ acting\ outwards\ perpendicularly.\right)$$

So the electrostatic pressure $P = \sigma^2/\varepsilon_0$.

1.10 Capacitors

A capacitor is a device, which can store electrical energy. The capacitance of a conductor is defined, as the charge required increasing its potential by unity. If q charge is required to increase the potential of a conductor by V then its capacitance, $C = \frac{q}{V}$, which is found to be independent of charge. The capacity of a conductor increases in the presence of neighbouring conductors due to induction of opposite charge at proximity and similar charge at relatively apart. It further increases if the neighbouring conductor is earthed. This is what is known to be 'Principle of capacitor'. Practically a capacitor is combination of two conductors with equal and opposite charges at so proximity that their potential difference remains unaffected for the presence of other charges but depends on the shape, size and proximity of two conductors and the intervening medium.

Its practical unit is farad. Capacitance of a capacitor is said to be 1-farad if 1C if charge, given to the unearthed plate increases the potenrtial difference between the plates by 1V.

1) The parallel plate capacitor

A set of two parallel conducting plates of same size with dielectrics or vacuum inside constitute a capacitor. To calulate its capacitance we take the separation to be very small compared to its lateral dimension, field inside the plates can be taken to uniform in between thr plates.

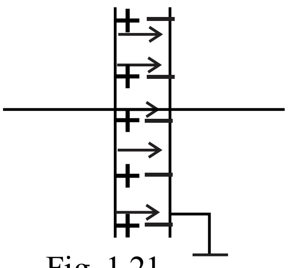


Fig. 1.21

The fig (1.21) shows a parallel plate capacitor of each plate area A separated by a distance d. Q be the charge given on plate-1, The plate-2 is earthed and is carrying the bound charge –q as in fig (1.20) V be the potential difference between the plates. σ be the charge per unit area of plate-1.

Then using Coluomb's theorem, the electric field inside the plates $E = \sigma/\varepsilon$ acting from plate-2 to plate-2 and it can be taken to be uniform considering d to be very small.

Thus the potential difference $V = E.d = \frac{\sigma d}{\varepsilon} = \frac{A\sigma d}{A\varepsilon} = \frac{qd}{A\varepsilon}$

Thus the capacitance of a parallel plate capacitor can be written as

$$C = \frac{q}{V} = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varepsilon_r A}{d} \quad \ldots\ldots(1.10.1)$$

Where ε_0 = permittivity of vacuum and ε_r = dielectric constant of intervening medium. In case of vacuum $\varepsilon_r = 1$. The eqn. Shows that capacitance is independent of charge.

2) Energy stored in a capacitor

Consider a capacitor has to be charged Q, then its potential be V. In the course of charging q be the charge on the one plate of capacitor and υ be its potential. Then for further charging through an infinitesimal charge dq the work done.

$$dW = vdq = \frac{qdq}{C}$$

So the total work done in charging the capacitor to the charge Q is

$$W = \int_0^Q \frac{qdq}{C} = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$ where V is the final potential difference between the plates.

... ... (1.10.3)

Thus the above expression gives the electrostatic energy stored in a capacitor.

3) Capacitors with layer of dielectrics

a) The dielectrics are parallel to plates :

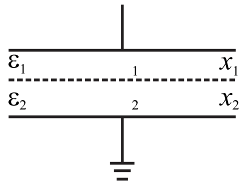


Fig. 1.22

The fig (1.3.1) shows a parallel plate capacitor filled with two dielectrics with dielectric constants ε_1 and ε_2, q be the charge given on the upper plate which raise its potential by V. E_1 and E_2 be the electrostatic field intensities in medim 1 and 2.

$$\text{Then } V = E_1x_1 + E_2x_2 = \frac{\sigma x_1}{\varepsilon_0\varepsilon_1} + \frac{\sigma}{\varepsilon_0\varepsilon_2}x_2 = \frac{\sigma}{\varepsilon_0\varepsilon_1}x_1 + \frac{\sigma}{\varepsilon_0\varepsilon_2}x_2 = \frac{q}{A\varepsilon_0}\left[\frac{x_1}{\varepsilon_1} + \frac{x_2}{\varepsilon_2}\right]$$

b) When the dielectrics are perpendicular to plates :

Here we consider the plate separation is d and the area portion A_1 and A_2 are occupied by the dielectris of dielectric constants ε_1 and ε_2 respectively. q be the charge given to the upper plate. The potential difference

$$V = E_1d + E_2d = \frac{\sigma}{\varepsilon_0\varepsilon_1}d + \frac{\sigma}{\varepsilon_0\varepsilon_2}d \quad \text{... ...} (1.10.4)$$

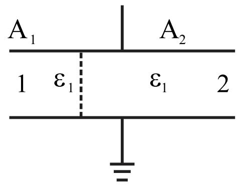


Fig. 1.23

(2) Combination of Capacitors

(a) Capacitors in series

The fig (1.24) shows a set of capacitors C_1, C_2, C_3....C_i....C_N, in series combination.

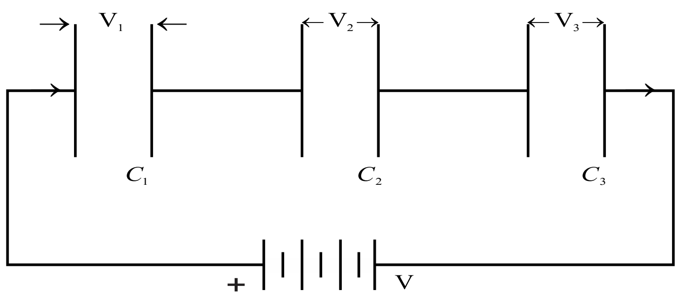


Fig. 1.24

To calculate the equivalent capacitance We apply a potential difference V across the combination. V_1, V_2, V_3.....V_i.....V_N be the Potential difference at steady state across C_1, C_2, C_3.....C_i.....C_i....C_N respectively. q be the charge on the + plate of the capacitors, it will be same for all plates as same current flows for the same time through each capacitor.

So, $V = V_1 + V_2 ..... = \frac{q}{C_1} + \frac{q}{C_2} .... = q\left[\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} .....\right]$

So the equivalent capacitance $C_{eq} = \frac{q}{V}$ *will give*

$$\frac{1}{C_{eq}} + \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ... = \sum_{i=1}^{i=N} \frac{1}{C_i}$$

(b) Capacitors in parallel

The fig (1.25) shows a set of capacitors C_1, C_2, C_3....C_i...C_N in parallel combination. N be the potential difference applied across the terminal of of the combination.

q_1, q_2............q_i...q_N bethe charge at steady state on capacitors C_1, C_2, C_3,.....C_N respectively.

The total charge $q = q_1 + q_2 + ... + q_i + ... + q_N$

$$= C_1V + C_2V + C_3V + ... + C_NV$$

If C_{eq} is the equivalent resistance of the combination then,

$q = C_{eq}V = C_1V + C_2V + C_3V + ... + C_NV$

Or, $C_{eq} = C_1 + C_2 + C_3 + ... + C_N$

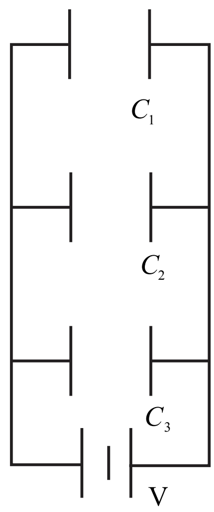


Fig. 1.25

(c) Energy loss due to sharing of charges of conductors

Cosider two conductors having charges q_1 and q_2, capacitances C_1 and C_2 and respective potentials V_1 and V_2 ($V_1 > V_2$). When they are make to touch or joined by a conducting wire, they will reach to common potential

$$V = \frac{Total\ Charge}{Commonpotential} = \frac{C_1V_1 + C_2V_2}{V_1 + V_2}$$

So the final charge on the conductors are

$$q_1' = C_1 V = C_1\left(\frac{C_1V_1 + C_2V_2}{C_1 + C_2}\right) \text{ and } q_2' = C_2 V = C_2\left(\frac{C_1V_1 + C_2V_2}{C_1 + C_2}\right)$$

So the initial electrostatic energy stored $E_i = \frac{1}{2}\left[C_1V_1^2 + C_2V_2^2\right]$

stored $E_f = \frac{1}{2}\left[C_1 + C_2\right]V^2 = \frac{1}{2}\left[C_1 + C_2\right]\left(\frac{C_1V_1 + C_2V_2}{C_1 + C_2}\right)^2$

So the change in energy due to sharing of charges

$$E_i - E_f = \Delta E = \frac{1}{2}\left[C_1V_1^2 + C_2V_2^2\right] - \frac{1}{2}\left[C_1 + C_2\right]\left(\frac{C_1V_1 + C_2V_2}{C_1 + C_2}\right)^2$$

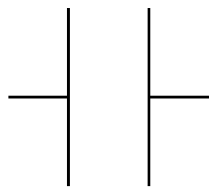
$$= \frac{C_1C_2}{2(C_1 + C_2)}(V_1 - V_2)^2 \text{ ... ...(1.10.5)}$$

The above equation shows ΔE to be always positive and thus there is always a loss of energy in sharing of charge.

(d) Force of attraction between plates of charged parallel pate capacitor

If σ is the surface charge density on one of the plates of parallel plate conductors then this charge is under the electric field σ/2ε for the charge on the other plate. So force on the unit area acting between the plates is $P = \sigma^2/2\varepsilon$. For the plate area A, the force is

$F = \frac{A\sigma^2}{2\varepsilon} = \frac{q^2}{2A\varepsilon}$ where q is the charge on one of the plate. This force in attractive as the plates bear opposite charges.



(e) Breakdown Voltage

A capacitor can withstand a limiting value of potential difference between its terminals, called break down voltage, which depends on the proximity of its plates, its structure and intervening dielectric. This is because, dielectric break down takes place at a limiting value of field intensity when the force on bonded electron crosses the bounding force of nucleus and the dielectric starts conducting. In case or air it is about 3×10^3 kV/m.

Example-1

A parallel plate capacitor with place separation d and plate area A each is filled with two dielectries of dielectric constants K_1 and k_2 in such a way that half the length of the plates is covered by each dielectric. Show that the capacitance is given by $C = \frac{\varepsilon_0 A}{2d}(k_1 + k_1)$.

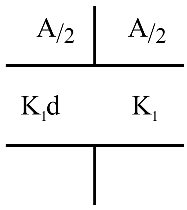


Fig. 1.26

Soln.

The fig. (1.26) shows such an arrangement that can be visualized as a parallel combination of two capacitances, $C_1 = \frac{\varepsilon_0 k_1 A}{2d}$ and $C_2 = \frac{\varepsilon_0 k_2 A}{2d}$

So the total Capacitance $C = C_1 + C_2 = \frac{\varepsilon_0 k_1 A}{2d} + \frac{\varepsilon_0 k_2 A}{2d} = \frac{\varepsilon_0 A}{2d}(k_1 + k_2)$ hence proved.

Example-2

Find the expression of force on a dielectric slab inserted partly in a parallel plate Capacitor maintained at constant potential difference Φ.

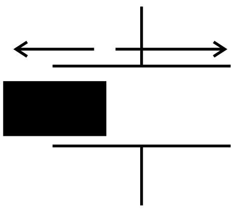


Fig. 1.27

The fig. 1.27 Shows a parallel plate capacitor of length *l and breath b.* d be the distance between the plates. A dielectric slab of breath b is inserted through a distance *x* as in fig. (1.27). The system can be considered as a parallel combination of two capacitors C_1 (portion with dielectric) & C_2 (portion without dielectric).

Thus $C = C_1 + C_2 = \frac{\varepsilon_0 kxb}{d} + \frac{\varepsilon_0 (l - x)}{d} = \frac{\varepsilon_0 b}{d}\left[kx + (1 - x)\right]$

So the energy stored in the system

$$U = \frac{1}{2}C\Phi^2 = \frac{1}{2}\frac{\varepsilon_0 b}{d}\left[kx + (1 - x)\right]\Phi^2$$

So the force acting on the dielectric slab

$$F = \frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\varepsilon_0 b}{d}\left\{kx + (1 - x)\right\}\Phi^2\right] = \frac{1}{2}\frac{\varepsilon_0 b}{d}(k - 1)\Phi^2$$

1.10 (a) Capacitance of Spherical Capacitor

(i) Isolated Sphere

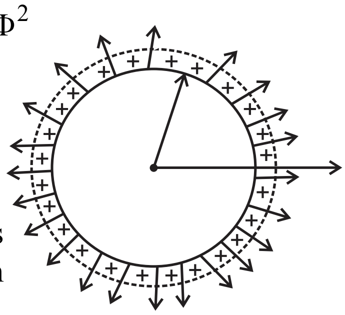


Fig. 1.28

The fig (1.28) shows an isolated conducting sphere of radius R. Q be the total charge on the surface. We consdier a gaussian surface given by dotted line. $\vec{E}$. be the electroic field intensity at a point on the surface. Then from Gauss's theorum $\int_S \vec{E}.d\vec{S} = Q/\varepsilon_0$

Or, $E4\pi r^2 = Q/\varepsilon_0$

$E = -\frac{dV_r}{dr} = \frac{Q}{4\pi\varepsilon_0}\frac{l}{r^2}$, where V_r is The potenhal on Gaussian surface.

So $dV_r = -\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}dr$

Considering V as the potential at R and the other plate to be at infinity, where the spotential is assumed to be zero we have

$$\int_V dV_r = -\frac{Q}{4\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$O - V = \frac{Q}{4\pi\varepsilon_0} = \frac{1}{r}\int_R^\infty = -\frac{Q}{4\pi\varepsilon_0}R$$

$$V = Q/_{4\pi\varepsilon_0} R$$

So $C = \frac{Q}{V} = 4\pi\varepsilon_0 R$ ... ...(1.10.10)

(ii) Two concentric spheres, outer sphere earthed

The fig (1.29) shows a spherical capacitor consiting of two concentric spherical conducting shells of iner radius a and other radius b.

Q be the charge given on the inner sphere, E be the intensity at point P, Through which passes a gaussian surface as shown dotted line. Then using Gauss's Theorem,

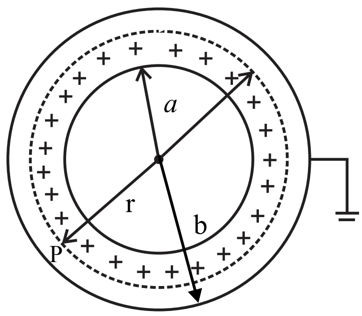


Fig. 1.29

$$\int_S \vec{E}.d\vec{S} = E4\pi r^2 = Q/\varepsilon_0$$

Or, $E = -\frac{dV}{dr} = \frac{Q}{4\pi\varepsilon_0 r^2}$

Or, $\int_{v}^{o} dv = -\int_a^b \frac{Q}{4\pi}\frac{dr}{\varepsilon r_2^{\,2}}$

$$-V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_a}^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{(b-a)}{ab}$$

So its capacittance $C = \frac{Q}{V} = 4\pi\varepsilon_0\frac{ab}{b-a}$ ... ...(1.10.7)

As $b \to \infty$ its gives the result of isolated capacitor.

(iii) Inner sphere earthed

Let q the charge given on the outer sphere of radius b. a be the radius of inner sphere (conducting) q' be the charge induced on inner sphere. Then from fig (1.30). As the potential on inner sphere is zero, Then $\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{b}+\frac{q'}{a}\right) = 0$

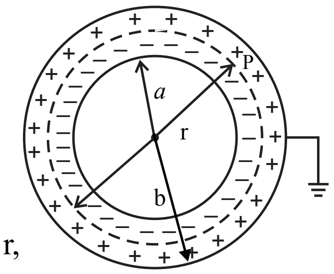


Fig. 1.30

Or, $q' = -\frac{a}{b}q$ ... ...(1.1.8)

Now considering the Gaussian surface Through P of radius r, We have using Gausse's theroem,

$$\int_S \vec{E}.d\vec{S} = q'/r = -\frac{a}{b}q$$

$$E = \frac{-a}{b}\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

Or, $\frac{dV}{dr} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ Or, $dV = \frac{1}{4\pi\varepsilon_0}\frac{a}{b}\frac{1}{r^2}dr$

Or, $\int_O^V dV = \frac{q}{4\pi\varepsilon_0}\frac{a}{b}\int_a^b \frac{dr}{r^2}$

$$V = \frac{q}{4\pi\varepsilon_0}\frac{a}{b}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{b-a}{b^2}\right)$$

$$C = \frac{q}{v} = 4\pi\varepsilon_0 \, b^2 \big/ (b-a) \quad ......(1.10.9)$$

(b) Capacitance of Cylindrical Capacitor.

Outer Cylinder earthed

The fig. (1.10.15) shows a cylindrical capacilor whose length L is much larger than its breadth so that electric field inside The cylinder can be considered to be axially symmeteric.

We consider a Gaussian surface a cylinder of radius r (r < r < b) as shown by dotted cylinder. $\vec{E}$ be the electric field at a point P on this Gaussian surface.

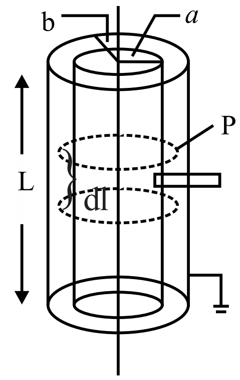


Fig. 1.31

Then applying Gauss's Theorem

$\int_S \vec{E}.d\vec{s} = \frac{q}{\varepsilon}$ (ε be The permittivity of medium in space between the cylinclers)

λ = charge/unit length

Or, $E 2\pi r \Delta l = \frac{\lambda \Delta l}{\varepsilon}$

$$E = \frac{\lambda}{2\pi\varepsilon}\frac{1}{r} = -\frac{dV}{dr}$$

$$dV = -\frac{1}{2\pi\varepsilon}\frac{dr}{r}$$

So potential difference between the cylinders

$$\int_O^V dV = -\frac{\lambda}{2\pi\varepsilon}\int_b^a \frac{dr}{r}$$

Or, $V = -\frac{\lambda}{2\pi\varepsilon}\ln a/b = \frac{\lambda}{2\pi\varepsilon}\ln b/a$

$= \frac{q}{2\pi\varepsilon L}\ln b/a$ (where $q = \lambda L$)

$$C = \frac{q}{V} = 2\pi\varepsilon L \Big/ \ln \frac{b}{a} \;\ldots\ldots(1.10.10)$$

(c) Capacitance of spherical capacitor with dielectrics

The fig (1.32) shows a spherical Capacitor of inner radius a and outer radius b with a dielectric of dieletric const K_1 Thickness t with inner sphere and The rest portion is filled in with dielectric constant K_2. q be the charge given on inner sphere Then potential difference between inner and outer sphere

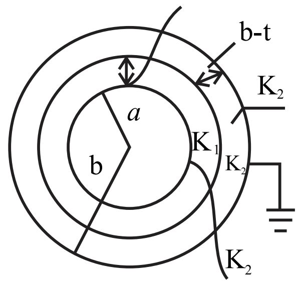


Fig. 1.32

$$V\cdot= \int_a^{a+t} \vec{E}_1 \cdot d\vec{r} + \int_{a+t}^{b} \vec{E}_1 \cdot d\vec{r} \;\ldots\ldots(1.10.11)$$

Where $\vec{E}_1$ and $\vec{E}_2$ are electric field intensity at The specific point in inner an outer sphere respectively

Now applying Gauss's Theorrein

$$\int_S \vec{E}_1 \cdot d\vec{S} = q \Big/ \varepsilon_0 k_1 \qquad \varepsilon_0 = \text{permittirty of free space.}$$

Or, $E_1 4\pi r^2 = \dfrac{q}{\varepsilon_0 k_1}$

Or, $\vec{E}_1 = \dfrac{q}{4\pi\varepsilon_0 k_1}\dfrac{1}{r^2}\hat{r} \;\; a < r \le a+t$

Similarly $\vec{E}_2 = \dfrac{q}{4\pi\varepsilon_0 k_2}\dfrac{1}{r^2}\hat{r} \;\; (a+t) \le r \le b$

So $\Delta V = \displaystyle\int_a^{a+t} \frac{q}{4\pi\varepsilon_0 k_1}\frac{1}{r^2}dr + \int_{a+t}^{b} \frac{q}{4\pi\varepsilon_0 k_2}\frac{1}{r^2}dr$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{k_1}\left(\frac{1}{a} - \frac{1}{a+t}\right) + \frac{1}{k_2}\left(-\frac{1}{b} + \frac{1}{a+t}\right)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{l}{k_1}\frac{t}{a(a+t)} + \frac{1}{k_2}\left\{\frac{b-(a+t)}{b(a+t)}\right\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{k_2bt + k_1a\{b-(a+t)\}}{k_1k_2ab(a+t)}\right]$$

$$\therefore C = \frac{q}{\Delta V} = \frac{4\pi\varepsilon_0 k_1k_2ab(a+t)}{k_2bt + k_1a(b-a-t)} \quad \ldots\ldots(1.10.12)$$

(d) Capacittance of Cylindrical Capacilor with dielectrics

The fig (1.33) shows a cylindercal capacitor consisting of two co-axial cylinder of radius a and b. (b > a). A dielectric of Thickness t and dicelctric conat k i is attached with inner cylinder and the rest portion is filled in with dielectric of dielectric constant k_2 as shown in fig (1.33) E_1 and E_2 be the electric field in first and second dielectric Then applying Gauss's Thorein as pervious $E_1 = \frac{\lambda\hat{r}}{2\pi\varepsilon_0 k_1 r}$ and $E_2 = \frac{\lambda\hat{r}}{2\pi\varepsilon_0 k_2 r}$ **

So the potential difference between inner and outer cylinder

** λ = charge/unit length on inner cylinder

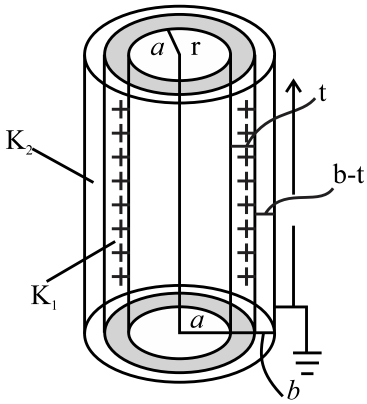


Fig. 1.33

$$\Delta V = \int_a^{a+t} \vec{E}_1\cdot d\vec{r} + \int_{a+t}^{b} \vec{E}_2\cdot d\vec{r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 k_1}\int_a^{a+t}\frac{dr}{r} + \frac{\lambda}{2\pi\varepsilon_0 k_2}\int_{a+t}^{b}\frac{dr}{r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{k_1}\ln\frac{a+t}{a} + \frac{1}{k_2}\ln\frac{b}{a+t}\right]$$

$$= \frac{q}{2\pi\varepsilon_0 L}\left[\frac{1}{k_1}\ln\frac{a+t}{a} + \frac{1}{k_2}\ln\frac{b}{a+t}\right]$$

$$\therefore C = \frac{q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\left[\frac{1}{k_1}\ln\frac{a+t}{a} + \frac{1}{k_2}\ln\frac{b}{a+t}\right]} \quad \ldots\ldots(1.10.13)$$

(c) Capacitance between two long parallel wires each of radius r separated by a distance d

The fig (1.34) shows such an arrangement where length of wire L > > a and d > > r the radius of wire.

Consider a point P at a distance at a distance x from wire A, λ be the charge per unit length of wire A and –λ be that of wire B.

The inlensity of field at P

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{x} + \frac{1}{d-x}\right]\hat{x}$$

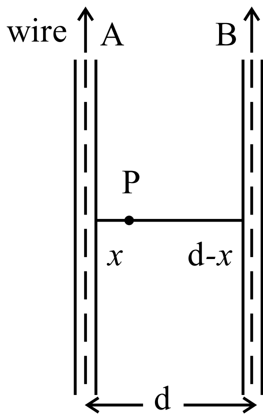


Fig. 1.34

So the potentrial difference between A and B

$$\Delta V = -\int_{d-r}^{r} \vec{E}\cdot d\vec{x} = \frac{\lambda}{2\pi\varepsilon_0}\int_{r}^{d-r}\left(\frac{1}{x} + \frac{1}{d-x}\right)dx$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln x - \ln(d-x)\right]^{d-r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{d-r}{r} - \ln\frac{r}{d-r}\right] = \frac{\lambda}{\pi\varepsilon_0}\ln\frac{d-r}{r}$$

Thus the capacitance per unit length

$$C = \frac{\lambda}{\Delta V} = \frac{\pi\varepsilon_0}{\ln\frac{d-r}{r}} \quad \ldots\ldots(1.10.14)$$

1.11 Electrical Image

Electrical imaging is a mathematical method of solving electrostatic problems introduced by Lord Kelvin. In this method the induced charge appeared on the surface (surfaces) due to the presence of point charge (charges) at proximity to the surface (surfaces) can be replaced by suitable point charge (charges) called image charge (charges), on one side of the surface. The field and potential on the other side of the surface can be evaluated by the superposition of field and potential due the charge (charges) and their image charge (charges).

Here also the Lapalce's equation is satisfied as with the actual carges and hence uniqueness theorem is followed.

Thus, electrical image (images) can be defined as a ficitious point charge (charges) on one side of a surface (surfaces) that can replace the effect of induced charge (charges) on other side of the surface (surfaces).

(1) Example–1

A point charge q is placed at (a, 0) in front of a infinitely extended earthed coducting sheet occupying (y, z) plane. Find

i) the position of image charge

ii) magnitude and nature of image charge

iii) Potential at a point x > 0

iv) field intensity on the surface of the sheet

v) force between the charge and the conductor.

vi) charge density at any point on the surface of the conductor.

vii) total induced charge on the surface.

Solution

The fig (1.35) shows a point charge q at position vector $\hat{i}a$ with respect to the origin O, in front of a large earthed coducting plane occupying y – z q_i plane, be the image charge as shown in fig. (1.35).

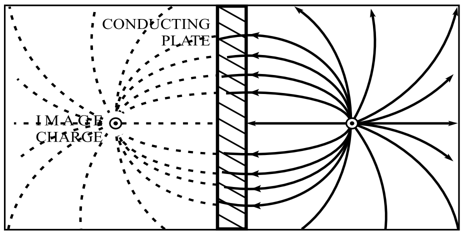


Fig. 1.35

i) From the symmerty of the field q_i lines is to be placed at $-\hat{i}a$

ii) From the condition of zero potential, at any point C on conducting plane

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{AC}+\frac{q_i}{BC}\right)=0, \text{ we have } q_i = -q$$

iii) The potential at point P (x, y, z), V = $V_0 + V_i$ = potential due to source charge q + potential due to image charge.

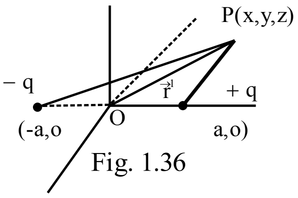


Fig. 1.36

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{r}-\hat{i}a\right|}-\frac{q}{\left|\vec{r}+\hat{i}a\right|}\right] \text{......}(1.11.1)$$

Where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ is the position vector of point P, $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along x, y and z axis.

Thus, $$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}}\right] \text{......}(1.11.2)$$

iv) The field intensity at point P (x, y, z) along x, y and z axis are

$$E_x = -\frac{\partial\Phi}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left[\frac{x-a}{\left\{(x-a)^2 + y^2 + z^2\right\}^{3/2}} - \frac{x+a}{\left\{(x+a)^2 + y^2 + z^2\right\}^{3/2}}\right]$$

$$E_y = -\frac{\partial\Phi}{\partial y}\left[\frac{y}{\left\{(x-a)^2 + y^2 + z^2\right\}^{3/2}} - \frac{y}{\left\{(x+a)^2 + y^2 + z^2\right\}^{3/2}}\right]$$

$$E_z = -\frac{\partial\Phi}{\partial y}\left[\frac{z}{\left\{(x-a)^2 + y^2 + z^2\right\}^{3/2}} - \frac{z}{\left\{(x+a)^2 + y^2 + z^2\right\}^{3/2}}\right]$$

On the surface of the conducting sheet x = 0, so only x-component of the field survive and $E_x = \dfrac{-2aq}{4\pi\varepsilon_0\left(a^2 + y^2 + z^2\right)^{3/2}}$ acting along –x axis. ... ... (11.11.3)

v) The induced charge can be replaced by the image charge –q fig (1.37). So the force betwee the surface and the charge q is $F = \dfrac{q^2}{4\pi\varepsilon_0 (2a)^2}$ and is attrative.

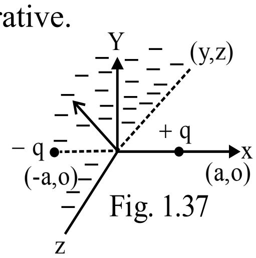


Fig. 1.37

v) σ be the surface density at a point (y, z) on the conductor face as in fig. (1.4.5)

Then

$$\sigma = -\varepsilon_0 E = \frac{-2aq}{4\pi\left(a^2+y^2+z^2\right)^{3/2}} = \frac{-aq}{2\pi\left(a^2+y^2+z^2\right)^{3/2}} \ldots (1.11.4)$$

vii) To calculate the total charge on the coductor we consider a ring of radius r and thickness dr about the origin 0 through the point (y, z).

Then the area of the elemental ring dA=2πrdr, charges on the ring = $\Delta 2\lambda$rdr

so the total charge is reduced.

$$\int_0^\infty \sigma 2\pi dr = \int_0^\infty \frac{-aq2\pi dr}{2\pi\left(a^2+y^2+z^3\right)^{3/2}} = \int_0^\infty \frac{-aqrdr}{2\pi\left(a^2+r^2\right)^{3/2}} = -q \ldots\ldots(1.11.5)$$

Problem–1

Find the work done in removing a charge q placed in front of earthed coductor at distance 'a' from the conductor.

Solution.

From the conception of electrical image, the induced charge can be replaced by –q charge on the other side of the surface at the same distance from the surface in which the charge q is present. The force of attraction between the charge q and the induced surface charge, when their distance of separation x is given by $F = \frac{q^2}{16\pi\varepsilon_0 x^2}$

Then the work to separate the charge to infinity is,

$W = \int_d^\infty \vec{F}.d\vec{r} = \frac{q^2}{16\pi\varepsilon_0 x^2}dx = \frac{q^2}{16\pi\varepsilon_0 d}$, which is the energy required for the separation.

1) Point charge placed between two large intersecting conducting sheet perpendicuar to each other.

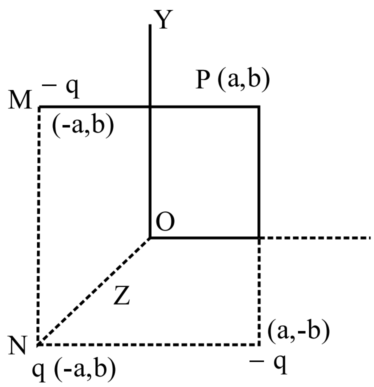


Fig. 1.38

The fig (1.38) shows a z = constant section of a large coducting plane occupying (x – z) and (y – z) plane.

Considering the conditions to be satisfied by the charges

1. Laplaces eqn. at all places excipt at point P ;

2. Potential over the conductor to be zero

3. Potential at infinity to be zero, Thus proceeding in the same way as previous for infinite earthed conducting plane, the image charges will be –q at (–a, b) ; + q at (–a, –b) and –q at (a, –b) as shown infig.

The potential at any point A (x, y)

$$V=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\hat{i}(x-a)+j(y-b)}-\frac{1}{\hat{i}(x+a)+j(y-b)}+\frac{1}{\hat{i}(x-a)+j(y-b)}-\frac{1}{\hat{i}(x-a)+\hat{j}(y-b)}\right]$$

The intensity $\vec{E}$ can be evaluated by eqn. $\vec{E}=-\nabla V.$ ... ...(1.11.5)

(2) Point charge in front of an earthed conducting sphere.

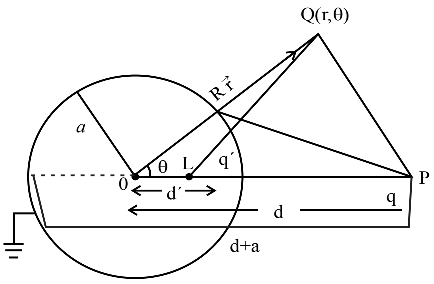


Fig. 1.39

Let a point charge is placed infornt of an eirthed conducting sphere of radius a.

Let the induced charges be replaced by image charge q' at a distane d' from centre O on the line OP due to symmetry of induced charge.

We must follow the conditions

i) Leplaces equation to be satisfied at all point r > a, except the point P

ii) Potential over surface of sphere be zero

iii) Potential at infinity from charges to be zero

Now from condition (ii), potentia on surface of sphere,

$$V_S=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{PR}+\frac{q'}{MR}\right]=0$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+d^2-2ad\cos\theta}}+\frac{q'}{\sqrt{a^2+d'^2-2ad'\cos\theta}}\right]=0$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q/a}{\sqrt{1+\frac{d^2}{a^2}-2\frac{d}{a}\cos\theta}}+\frac{q'/a}{\sqrt{1+\frac{a^2}{d'^2}-\frac{2a}{d'}\cos\theta}}\right]=0$$

As this equation is valid for all values of θ, we can write $q'/d' = -q/a \;\&\; \frac{d}{a} = a/d'$

Or, $d' = a^2/d \;\&\; q' = q\frac{d'}{a} = -\frac{qa}{d}$ ... ...(1.11.7)

Thus we know the position, quantity and sign of image charge.

Potential at point Q.

$$V_Q = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}}-\frac{a/d}{\sqrt{r^2+d'^2-2rd'\cos\theta}}\right]$$

Or, $$V_Q = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}}-\frac{a}{\sqrt{r^2d^2+a^4-2rda^2\cos\theta}}\right]$$

$E_r = \frac{\partial V_Q}{\partial r}$ and $E_\theta = -\frac{1}{rd\theta}\frac{\partial V_Q}{\partial\theta}$ can be calculated ... ... (1.1.8)

Or, $$V_Q = \frac{q}{4\pi\varepsilon_0}\left[\left(r^2+d^2-2rd\cos\theta\right)^{-1/2}-\left(a^2+r^2\frac{d^2}{a^2}-2rd\cos\theta\right)^{-1/2}\right]$$

when r = a i.e. wehen, Q is a point on the surface of the sphere, then V_Q = O

In order to find the The surface density on the surface of the sphere, we must know The radial component of the electric field.

$$E_r = -\left.\frac{\partial V_Q}{\partial r}\right|_\theta = \frac{q}{4\pi\varepsilon_0}$$

$$\left[\frac{r-d\cos\theta}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}-\frac{\frac{d^2r}{a^2}-d\cos\theta}{\left(a^2+\frac{d^2r^2}{a^2}-2rd\cos\theta\right)^{3/2}}\right] \ldots\ldots(1.11.10)$$

when the Q point is on the surface of the sphere,

when r = a, then

$$\left(E_r\right)_{r=a}=-\frac{q}{4\pi\varepsilon_0}\frac{\frac{d^2}{a}-a}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}$$

$$=\frac{q}{4\pi\varepsilon_0 l^3}\left(a-d^2/a\right)$$

where ℓ = RP = $\left(a^2+d^2-2ad\cos\theta\right)^{1/2}$

Or, we can write

$$E_n=\left(E_r\right)_{r=a}=\frac{q}{4\pi\varepsilon_0 l^3}\left[a-d^2/a\right] \ldots\ldots(1.11.12)$$

Induced charge density at on the sphere : If σ be the induced surface charge density at the point R on the spare.

$$\sigma=\varepsilon_0 E_n=\frac{q}{4\pi\ell^3}\left(a-d^2/a\right)$$

$$=-\frac{q}{4\pi l^3}\left[\frac{d^2-a^2}{a}\right] \ldots\ldots(1.11.13)$$

Since d > a, equation (1.11.13) shows that σ is negative. The magnitude of the surface charge density is maximum when θ = O, i.e., l is minimum, and minimum when $\theta = \pi$.

The ratio of manimum and the minimum surface charge density is given by,

$$\left(\frac{l_{man}}{l_{\min}}\right)^3 = \left(\frac{d+a}{d-a}\right)^3$$

Force exerted on the point charge + q at P.

which is altractive,

$$F = -\frac{qq'}{4\pi\varepsilon_0 LP^2} = \frac{q(-q\,a/d)}{4\pi\varepsilon_0\left(d-a^2/d\right)^2}$$

$$= \frac{-q^2ad}{4\pi\varepsilon_0\left(d^2-a^2\right)^2} \;\ldots\ldots(1.11.14)$$

(3) Point charge in front of an unearthed condcting sphere.

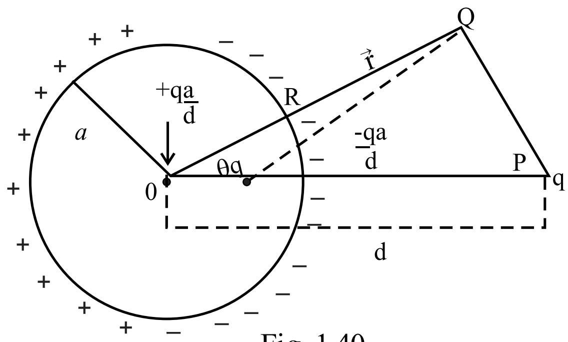


Fig. 1.40

The fig (1.40) show an unearthed conducting sphere of radius a. Here to calculate the image charge and its position the following conditions must be satisfied :

i) Laplaces equation for r > a except the point P,

ii) Total charge induced is zero.

iii) Sphere surface is at constant potential.

iv) Potential at infinity is zero.

So, this case will be similar to the earthed conducting sphere condition with a charge $+q\frac{a}{d}$ is to be placed at the centre of conducting sphere to satisfy the condition (ii) as mentioned in fig (Fig.1.40).

So the surface potential now will be $V_S = \frac{1}{4\pi\varepsilon_0}\frac{q\,a/d}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q}{d}$ instead of zero.

The potential at point Q will be

$$V_Q = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{a}{\sqrt{r^2d^2+a^4-2ra^2d\cos\theta}} + \frac{a}{rd}\right] \text{......}(1.11.4)$$

The radial component of field at $Q(r,\theta)$

$$E_r = -\frac{\partial V_Q}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[\frac{r-d\cos\theta}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}} - \frac{ad\left(rd-a^2\cos\theta\right)}{\left(r^2d^2+a^4-2ra^2d\cos\theta\right)^{3/2}}\right.$$

$$\left. + a/r^2d\right] \text{......}(1.11.5)$$

On the surface of sphere (r = a)

$$E_n = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}\left(a-\frac{d^2}{a}\right) + \frac{1}{ad}\right] \text{......}(1.11.15)$$

So the induced charge per unit area on the surface,

$\sigma = \varepsilon_0 E_n$ will be maximum for $\theta = 0$. .... (1.11.16)

$$\sigma = -\frac{q}{4\pi a}\left[\frac{d^2-a^2}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}} - \frac{1}{d}\right]$$

Let us see the nature of nature of the induced charge, whether positive or negative when $\theta = 0$

$$\sigma = -\frac{q}{4\pi a}\left[\frac{d^2-a^2}{(d-a)^3} - \frac{1}{d}\right]$$

$$= -\frac{q}{4\pi a}\frac{3ad-a^2}{(d-a)^2}$$

Which is clearly negative. Now when $\theta = \pi$

$$\sigma = -\frac{q}{4\pi a}\left[\frac{d^2 - a^2}{(d+a)^3} - \frac{1}{d}\right]$$

$$= \frac{q}{4\pi d_a}\left[\frac{a+3d}{(d+a)^2}\right]$$

Which is σ positive charge density

There is no induced charges on the sphere separating the regions of positive and negative charge density : NOW, putting $\sigma = 0$ in eqaution, we have for the line of non-electrification.

$$Cos\,\theta_N = \frac{a^2 + d^2 - \left[d\left(d^2 - a^2\right)\right]^{2/3}}{2ad}$$

Which is positive, it implies that $\theta_N < \pi/2$

Force experienced by the charge at P

$$F = \frac{q^2{}_{ad}}{4\pi\varepsilon_0}\left(d^2 - a^2\right)^2 + \frac{q^2 a}{4\pi\varepsilon_0 d^3} = -\frac{q^2}{4\pi\varepsilon_0}\left(\frac{a}{d}\right)^3 \frac{2d^2 - a^2}{\left(d^2 - a^2\right)^2}$$

1.12 Summary

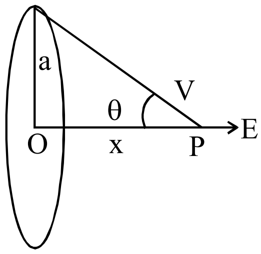
After studying the unit we should understand following :

1. Interaction between two charges through Coulomb's law and for a cluster of charges.

2. Conception of electric field electric field $\vec{E}$ lines. Conservative nature of $\vec{E}$ and introduction of electric potential V through $\vec{E} = -\Delta V$.

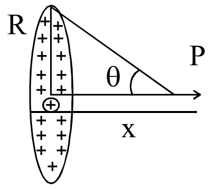
(a) Point charge $V = \int_{\infty}^{r} -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ and $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

(b) due to a uniformly charged ring on its axis at a distance x from its centre α be The charge per unit length.



$$V=\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{a^2+x^2}},\quad \overrightarrow{E_P}=\frac{1}{4\pi\varepsilon_0}\frac{q\vec{x}}{\left(a^2+x^2\right)^{3/2}},\quad q=2\pi a\alpha.$$

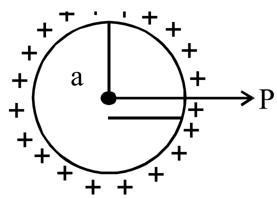
(c) due to a charged disc (σ = charge per unit area)



$$V_P=\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+x^2}-x\right];\ \vec{E}_P=\frac{q}{2\pi\varepsilon_0 R^2}(1-\cos\theta)\hat{x}$$

(d) due to spherical distribution of charge

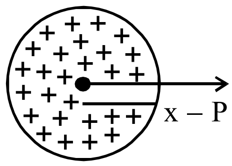
(i) Uniformly charge spherical shell.



(a) $x<a_1 V_i=\frac{1}{4\pi\varepsilon_0}q/a,\quad E_i=0$

(b) $x>aV_0=\frac{1}{4\pi\varepsilon_0}\frac{q}{x},\quad E_0=\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}\hat{x}$

(ii) Uniformly charge sphere. P = charge density



$$E_i=\frac{1}{4\pi\varepsilon_0}=\frac{4/3\,\pi x^3 p\hat{x}}{x^2},\ E_0=\frac{1}{4\pi\varepsilon_0}\frac{q\,\hat{x}}{x^2}$$

Gausse's law and its and its application for various distribution of charge.

Laplace and Poission's equations

Poission's eqn $\nabla^2\phi=P/\varepsilon_0$

Laplace's eqn $\nabla^2\phi=0$ and their one dimensional soluns.

Eletrostatic energy

$$U=\frac{1}{2}\sum\frac{1}{4\pi\varepsilon_0}\frac{qiqj}{rij};\ U=\frac{1}{2}\int_V\vec{E}\cdot\vec{D}\,dv$$

Study of capacitors

Capacitance of parallel plate capacitor $C=\varepsilon_0 A\Big/\sum x/\varepsilon$

Capacitors in series $(1/c\,eq)=\sum 1/c$

Capacitors in parallel $C_{eq} = \sum c$

Energy loss in capacitor $= \frac{1}{2}\frac{c_1 c_2}{c_1 + c_2}(v_1 - v_2)^2$ due to sharing of charge

4. Capacitance of spherical apacitor $c = 4\pi\varepsilon_0 R$

Two concentric spheres, outer sphere grounded $c = 4\pi\varepsilon_0 \frac{ab}{b-a}$

Capactance of cylinder $c = 2\pi\varepsilon L/\ln/b/a$

Spherical capacitor with dieteclric $c = \frac{4\pi\varepsilon_0 k_1 k_2 (a+t) k_1}{k_2 bt + k_1 a(b-a-t)}$

Capacitance of cylinderical capactor with dielectric, $c = \frac{2\pi\varepsilon_0 L}{\left[\frac{1}{k_1}\ln\frac{a+t}{a} + \frac{1}{k_2}\ln\frac{b}{a+t}\right]}$

Capacitance of two parallel wires, $c = \frac{\pi\varepsilon_0}{\ln\frac{d-r}{r}}$

5. **Electrical image**

Point charge in-fornt of a earthed unifinite cmducting plane.

Point charge infornt of earhed conducting sphere

Point charge in front of unearthed conducting sphere

1.13 Review question and answer

QNO 1 Why the electric field inside a good condutor is zero in a steady state and any net charge on a good conductor must be entirdy on the surface?

Answer : If there were field, charges would move, charges will move until they find the arrangement that makes the eletric field zero in the interior. if therby where charge in the interior, Then by Gaussn law there would be a field in the interior, which cannot be true.

QNO 2 Why do electric field lines never cross each other?

Answer : It is so because if they cross each other then at the point of interessection there will be two tangent's which is not possible.

QNO 3 What is the net amount of charge on a charged capacitor?

Answer : The net charge of a charge capactior is zero because the charge on its two plates are equal number and opposite in sign. Even when the capacitor is discharged net charge on the capacitor remains zero because each plate has zero charge.

QNO 4. How does the field line and an equapotential surface behave?

Answer : They are always at 90º

QNO 5. What is the power dissipoted in a pure capacitor?

Answer : Zero

QNO 6. What will be the potential difference between the plates when a dielctric slab is introduced in parallel plate capacitor?

Answer : decrease.

QNO 7. A point charge q is held at a distance d infornt of an infinite grouded conducting plane what is the electric potential infornt of the plane?

Answer : See Article 1.11. for answer.

QNO 8. A point charge q is placed at a distance d from the centre of a grounded conducting sphere of radius a (a < d). Calculate the density of the induced surface charge on the sphere?

Answer : See article 1.11 for answer.

QNO 9. The concentric spheres of radii r_1 and r_2 ($r_1 > r_2$) carry electric charges + Q and –Q respectively. The region between The plates is filled with two insulating layors of dielectric constant ε_1 and ε_2 with widths d_1 and d_2 respectivly. Compute the capacitane of the system

Answer : See article 1.10 C for answer.

QNO 10. Which one of the following is an impossible in electrostatic field?–

i) $\vec{E} = xy\hat{i} + (2yz)\hat{j} + (3xz)\hat{k}$

ii) $\vec{E} = y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + (2yz)\hat{k}$

Answer : For solution, if $\vec{\nabla} \times \vec{E} = 0$, Then That electric field exists in electrostatics correct answer is (1)

QNO 11. There charges Q, + q and + q are placed at the vertices of right angled isoscles traingle as shown in the figure. What is the value of electrostatic energy?

Solution : Length of the hypotenase $= \sqrt{2a}$, the net electro static energy is–

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{Qq}{a} + \frac{Qq}{\sqrt{2}a} + \frac{qq}{a}\right]$$

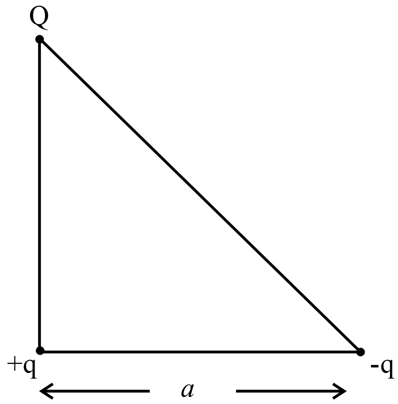


Fig. 1.41

QNO 12. Three infinite long plane sheets carrying unfirom charge densitics $\sigma_1 = \sigma$, $\sigma_2 = +2\sigma$, and $\sigma_3 = +3\sigma$ are placed parallel to × z plane at z = a, z = 3a, and z = 4a as shown in the figure.... what is value of electric field at the point Q?

Solution : The electric field a point Q due to an infinite long plane sheet carrying uniform charge density is given by–

$$E = \frac{\sigma}{2\varepsilon_0}$$

which is independent of the distance of point Q from the sheet and is, therefore uniform The direction of the electric field. is away from the sheet and prependicular to it if σ is positve and it towards the sheet and perpendicuar to it if σ is negative so

$\vec{E}_1 = \frac{\sigma}{2\varepsilon_0}\left(-\hat{k}\right)$ along – ve z direction

$\vec{E}_1 = \frac{2\sigma}{2\varepsilon_0}\left(-\hat{k}\right)$ along –ve z direction

and $E_3 = \frac{3\sigma}{2\varepsilon_0}\left(-\hat{k}\right)$ along – ve z direction

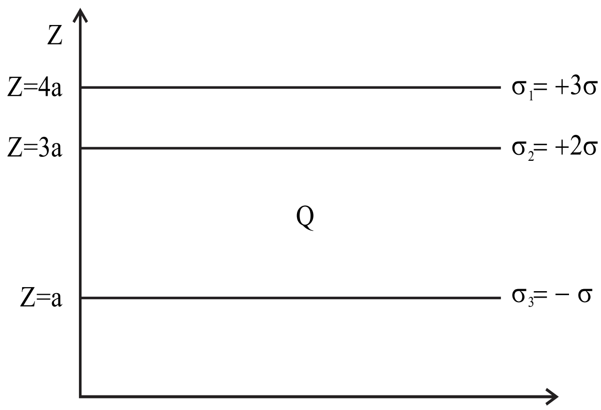


Fig.1.42

From the superposition principle. The resultant electric field at point Q is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$=\frac{\sigma}{2\varepsilon_0}\left(-\hat{k}\right)+\frac{2\sigma}{2\varepsilon_0}\left(-\hat{k}\right)+\frac{3\sigma}{2\varepsilon_0}\left(-\hat{k}\right)$$

$$=-\frac{3\sigma}{\varepsilon_0}\hat{k}$$

1.14 Problems & Solutions

QNO 1. The electric field in a certain region is given as $\vec{E}=Ar^3\hat{r}.$ Prove that charge contained within a spherical surface of radius 'a' centred at the origin is $4\pi\varepsilon_0 Aa^5$.

Solution : From The differential form of Gauss's in law $\vec{\nabla}\cdot\vec{E}=\rho/\varepsilon_0$

So the charge density $\rho=\varepsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2Ar^3\right)\right]$

(using spherical polar co-ordinate)

$$\rho=5\varepsilon_0 Ar^2$$

Total charge within a sphere of radius 'a' is $Q=\int_0^a\rho(r)4\pi r^2dr$

$$=20\pi\varepsilon_0 A\int_0^a r^4dr=4\pi\varepsilon_0 Aa^5$$

QNO 2. The electrostatic potential due to a charge distribution is given by $V(r)=\frac{q}{4\pi\varepsilon_0}=\frac{e^{-r/\lambda}}{r}$ enclosed within a sphere of radius $1/\lambda$ given by $\frac{2q}{e}$.

Solution : Given $V(r)=\frac{q}{4\pi\varepsilon_0}\frac{e^{-\lambda r}}{r}$

So the electrical field is

$$\vec{E}=-\vec{\nabla}V=-\frac{q}{4\pi\varepsilon_0}\left[\frac{re^{-\lambda r}(-\lambda)-e^{-\lambda r}}{r^2}\right]\hat{r}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{e^{-\lambda r}}{r^2} (1 + \lambda r) \hat{r}$$

So the total charge enclosed within a sphere of radius r is

$$Q_{encl} = \varepsilon_0 \oint \vec{E} . d\vec{s}$$

$$= \frac{\varepsilon_0 q}{4\pi\varepsilon_0} \int_0^{\pi} \int_0^{2\pi} \frac{e^{-\lambda r}}{r^2} (1 + \lambda r) \hat{r} \cdot \sin^2 d\theta d\phi dr\hat{r}$$

$$= qe^{-\lambda r} (1 + \lambda r)$$

Thus the total charge enclosed within a sphere of radius $r = \frac{1}{\lambda}$ is

$$Q_{encl} = Qe^{-\lambda/\lambda} \left(1 + \frac{1}{\lambda} \lambda\right)$$

$$= 2 q/e$$

QNO 3. Two large non-conducting sheet one with a fixed uniform positive charge and another with a fixed uniform, negative charge are placed a distance of 1 meter from each other. The magnitude of the surface charge densities are $\sigma_+ = 10 \mu c / m^2$ and $\sigma_- = 5 \mu c / m^2$

for the positive plate and negative plate respectively. What is the electric field in the between the sheets?

Solution : The electrical field between the sheet is

$$= \frac{\sigma_+}{2\varepsilon_0} + \frac{\sigma_e}{2\varepsilon_0}$$

$$= \frac{1}{2\varepsilon_0} [10 + 5] \times 10^{-6}$$

$$= \frac{15 \times 10^{-6}}{2 \times 8{\cdot}86 \times 10^{-12}}$$

$$= {\cdot}8465 \times 10^{6}$$

$$= 8{\cdot}46 \times 10^{5}\ N/C$$

QNO 4. A long cylinder carries a charge density ρ = Ar ; for some constant A. Find the electric field inside the cylinder.

Solution : We draw Gaussian cylinder of length l, and radius r. For this surface, Gauss's law state that

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{1}{\varepsilon_0} Q_{encle}$$

So the enclosed charge is

$$Q_{encl} = \int \rho\, dV = \int_0^e \int_0^{2\pi} \int_0^r (Ar)(rdr)\, d\phi dz$$

$$= 2\pi Al \int_0^r r^2 dr = \frac{2}{3}\pi Alr^3$$

Now from symmetry, it is cleart that $\vec{E}$ is radially out word, so for the curved portion for the Gaussian surface we get

$$\int \vec{E}\cdot\vec{ds} = |E| \int ds = |E| 2\pi rl$$

Hence $\left|\vec{E}\right| 2\pi rl = \frac{2}{3\varepsilon_0}\pi Ar^3$

So finally $\vec{E} = \frac{1}{3\varepsilon_0} Ar^2 \hat{r}$

QNO 5. A hollow, conducting sphereial shell of inner radius 'a' and outer radius be encloses a charge –'q' inisde, which is located at a distance d < a from The centre of the sphere what is the potential of the centre of the shell?

Solution : charge induced on the inner sphere is + q and charge induced on the outer sphere is + q

Thus the potential at the centre of the sphereical shell is

$$V = \frac{1}{4\pi\varepsilon_0}\left[-\frac{q}{d}+\frac{q}{a}-\frac{q}{b}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{q}{a}-\frac{q}{d}-\frac{q}{b}\right]$$

QNO 6. Given the potential $V = \frac{5}{r}\sin\theta\cos\phi$

(a) Find the electric flun density $\vec{D}$ at $\left(1, \pi/2\, 0\right)$

(b) Calculate the work done in maving a $100\,\mu c$ charge from a point (2, 30º, 120º) to B (4, 90º, 60º)

Solution : $\vec{D} = \varepsilon_0 \vec{E}$

But $\vec{E} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}\right]$

$$= 5\left[\frac{\sin\theta\cos\phi\hat{r}}{r^2}-\frac{1}{r^2}\cos\theta\cos\phi+\frac{1}{r^2}\sin\theta\hat{\phi}\right]$$

At $\left(1, \pi/2, 0\right)$

$$\vec{D} = \varepsilon_0\vec{E} = \varepsilon_0 5\left[\hat{r}-0\hat{\theta}+0\hat{\phi}\right]$$

$$= 5\times 8{\cdot}85\times 10^{-12}\, c/m^2$$

$$= 44{\cdot}25\, c/m^2$$

work done $W = -Q\int_A^B \vec{E}.\vec{dl} = Q.V_{AB}$

$$= Q\left(V_B - V_A\right)$$

$$= 100\left[\frac{5}{4}\times\frac{1}{2}-\frac{-5}{2\times 2}+\frac{1}{2}\right]\times 10^{-6}$$

$= 125\mu J$

QNO 7. Consider two electric dipoles with their centres at fined distance of reparation. Show that if the angles of the dipoles make with the line joining ther centres are θ_1 and θ_2 and if θ_1 is held fined Then for equillibrium $\tan\theta_1 + 2\tan\theta_2 = 0$

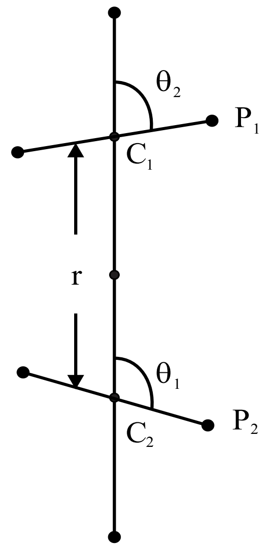


Fig.1.49

Solution : Configuration of the dipoles are show in the figure.

We know that interaction energy between two dipoles,

$$W_{21} = \frac{P_2 P_1}{r^3}(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2)$$

The couple exerted by the first dipole on the second is

$$\tau_{21} = \frac{\partial W_{21}}{\partial\theta_2} = \frac{P_1 P_2}{r^3}(\sin\theta_1 \cos\theta_2 + 2\cos\theta_1 \sin\theta_2)$$

At equilibrium $\tau_{21} = 0$

$\therefore\ \sin\theta_1 \cos_2 + 2\cos\theta_1 \sin\theta_2 = 0$

$\Rightarrow\ \tan\theta_1 + 2\tan\theta_2 = 0$ Proved

QNO 8. Three electric charges are placed at the corners of an equilateral traingle ΔABC of length 'a'. Find the magnitude of the dipole of the system.

Solution : Taking B as the origin of two dimensional co-ordinate system B (O, O) $A\left(a/2\frac{\sqrt{3}}{2}\right)$ and $c(a,0)$

The dipole moment the system as in equilateral traingle ΔABC,

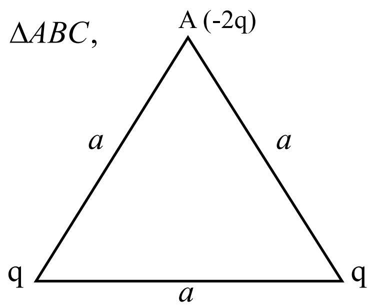


Fig.1.50

$$\vec{P} = aq\hat{l} + \left[\frac{a}{2}\hat{i} + \frac{\sqrt{3}a\hat{j}}{2}\right](-2q)$$

$$= -\sqrt{3}aq\,\hat{j}$$

The magnitude of the dipole is $= \sqrt{3}qa$

QNO 9. A sphere of radius R centred at the origin carries charge density $\rho(r,\theta), = k \cdot \frac{R}{r^2}(R - 2r)\sin\theta$ where k is a constant, and $\rho(r,\theta)$ are usual spherical co-ordinales. Find the approrimate potential for points on z axis far from the sphere.

Solution : Monopole terme for potential

$$V_{monopole} V_1 \alpha \int \rho dV = \int_{\theta=0}^{R} \int_{0}^{\phi=2\pi} \rho(r',\theta) r'^2 \sin\theta d\theta\phi$$

Dipole term $V_2 = \int r' \cos\theta s dV = 0$

and Qudruipole term $V_3 = \frac{1}{4\pi\varepsilon_0 r^3} \int r'^2 \frac{1}{2}(3\cos 2\theta - 1)\rho dV$

$$\approx \frac{1}{4\pi\varepsilon_0} \frac{k\pi R^5}{48z^3}(r \to z)$$

QNO 10. Find the capacity of concertric spheres with two dielectric.

Solution : Let Q charge given on the inner sphere, so the electric field within the radius a to be

$$E_1 = \frac{Q}{4\pi\varepsilon_0 \varepsilon_{1r} r^2}$$

So the potential difference across ε_1

$$V_1 = -\int_a^b \vec{E} \cdot \vec{dr} = -\frac{Q}{4\pi\varepsilon_0 \cdot \varepsilon_{1r}} \int_a^b \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 \varepsilon_{1r}} \left[\frac{1}{a} - 1/b\right]$$

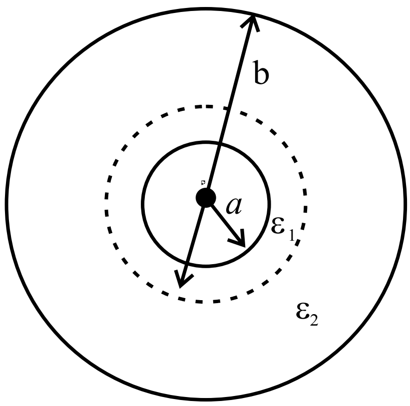


Fig.1.51

Similary the potential difference across ε_2

$$V_2 = \int_b^c \frac{Q}{4\pi\varepsilon_0 \varepsilon_{2r}} \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0\varepsilon_{20}}\left[\frac{1}{b} - \frac{1}{c}\right]$$

Total PD V = V_1 + V_2

Capacitance $C = \dfrac{Q}{V_1 + V_2}$

$$= \frac{4\pi\varepsilon_0}{\frac{1}{\varepsilon_{1r}}\left[\frac{1}{a} - \frac{1}{b}\right] + \frac{1}{\varepsilon_{2r}}\left[\frac{1}{b} - \frac{1}{c}\right]}$$

Alternative method :

Using spherical co-ordinate system, to find but the potential in at any point inside the dielectric, taking note that $\dfrac{\partial V}{\partial \theta} = 0$ and, $\dfrac{\partial V}{\partial \phi} = 0$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)\right] = 0$$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = 0 \Rightarrow r^2 \frac{\partial V}{\partial r} = c_1$$

where c_1 is constant of integration

Again intergating, in region I

$$V_1 = -\frac{c_1}{r} + c_2 \text{ in } \varepsilon_1$$

$$V_2 = -\frac{C_1'}{r} + c_2' \text{ in } \varepsilon_2$$

Then from, $\vec{E}, -grad\ v_1 \hat{r}$ we get

$$\vec{E}_1 = -\hat{r}\frac{c_1}{r^2},\ a < r < b$$

$$\vec{E}_2 = -\hat{r}\frac{c_1'}{r^2},\ b < r < c$$

$$\vec{D}_1 = -\hat{r}\varepsilon_0\varepsilon_{1r}\frac{c_1}{r^2} \quad a < r < b$$

$$\vec{D}_2 = -\hat{r}\varepsilon_0\varepsilon_{2r}\frac{c_1'}{r^2} \quad b < r < c$$

We observe that there are four unkowns, so we need four boundary conditions, to evaluate therm, They are,

on $r = a, V_1 = V_a = \frac{c_1}{a} + c_2$

on $r = c, V_2 = V_c = -\frac{c_1'}{c} + c_2'$

on $r = b, V_1 = V_2$; i.e. $-\frac{c_1}{b} + c_2 = -\frac{c_1'}{b} + c_2'$

on r = b, D_n is continouons, so, $D_{nl} = D_{nz}$

or, $D_{rl} = D_{r2}$ i.e. $\varepsilon_0\varepsilon_{r1}\frac{c_1}{b^2} = \varepsilon_0\varepsilon_{2r}\frac{c_1'}{b^2}$

From, these boundary conditions, we obtain 'D' on the surface of the inner sphere of radius a

$$D_1 = \frac{\varepsilon_0\left(V_c - V_a\right)}{\left[\frac{1}{\varepsilon_{2r}}\left(\frac{1}{b} - 1/c\right) + \frac{1}{\varepsilon_{1r}}\left(\frac{1}{a} - 1/b\right)\right]}\frac{1}{a^2}$$

So the total charge on the inner sphere, will be $4\pi a^2$ times D on 'a'

$$\therefore \text{ Capctance } C = \frac{charge}{V_c - V_a}$$

$$= \frac{4\pi\varepsilon_0}{\frac{1}{\varepsilon_2 r}\left(\frac{1}{b} - 1/c\right) + \frac{1}{\varepsilon_1 r}\left(\frac{1}{a} - 1/b\right)}$$

QNO 11. A dipole having a moment $\left(3\hat{i}-5\hat{j}+10\hat{k}\right)$ me m is loated at Q (1, 2, –4) in free space. Find V at P (3, 3, 4)

Solution : Unit vector along the straight line $PQ \cdot \hat{r} = \frac{2i+\hat{j}+8\hat{k}}{\sqrt{69}}$

Potential at the point P (3, 3, 4)

$$= \frac{\vec{P}\cdot\hat{r}\times 10^{-9}}{4\pi\varepsilon_0 r^2} = \frac{\left(3\hat{i}-5j+10k\right)\cdot\left(2i+j+8k\right)\times 10^{-9}}{\sqrt{69}\left(69\right)}$$

$$= 1{\cdot}27\, V/m$$

QNO 12. Three point charges are located as shown in the figure Fig (1.52) Find the approximate electric field at points far from the origin state your answer is spherical co-ordinates, and include The lowest orders in multipote expansion.

Solution : Total charges Q = 3q – q – q = q

$V_{mono} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ and dipole moment $p = 3qa\hat{z}$

so, $V_{dip} = \frac{3qa\cos\theta}{4\pi\varepsilon_0 r^2}$ Therefore

$$V\left(r,\theta\right) \cong \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}+\frac{3a\cos\theta}{r^2}\right]$$

$$\vec{E}\left(r,\theta\right) \cong \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r^2}\hat{r}+\frac{3a}{r^3}\right]\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)$$

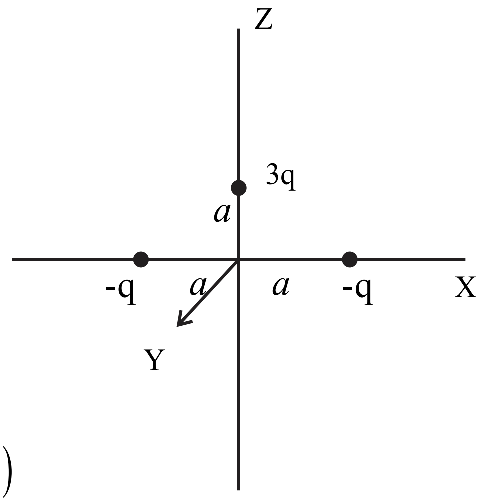


Fig.1.52

QNO 13 Consider an electric dipole $\vec{P}$, which is is fined at a distance zo along the z-ams and at an orientation θ with respect to that axis, consider the xy plane as conductor at zero potential. what is the charge density on the conductor induced by the dipole

Solution : As shown in the figure the dipole is $\vec{P} = P\left(\sin\theta, 0, \cos\theta\right)$ and its image dipole is $\vec{P}' = P\left(-\sin\theta, 0, \cos\theta\right)$ In the region z > 0, The potential at a point $\vec{r}\left(x, yz\right)$

is $V(\vec{r}) = \frac{1}{4\varepsilon_0}\left[\frac{Px\sin\theta + (z - z_0)\cos\theta}{x^2 + y^2 + (z - z_0)^2}\right]^{3/2} + P\left[\frac{\{-x\sin\theta + (z + z_0)\cos\theta\}}{\{x^2 + y^2 + (z + z_0)^2\}^{3/2}}\right]$

The induced charge density on the surface of the coductor is given by

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z = z_0}$$

$$= \frac{P\cos\theta}{2\pi\left(x^2 + y^2 + z_0^2\right)^{3/2}}$$

$$+ \frac{3Pz_0\left(-x\sin\theta + z_0\cos\theta\right)}{2\pi\left(x^2 + y^2 + z_0^2\right)^{5/2}}$$

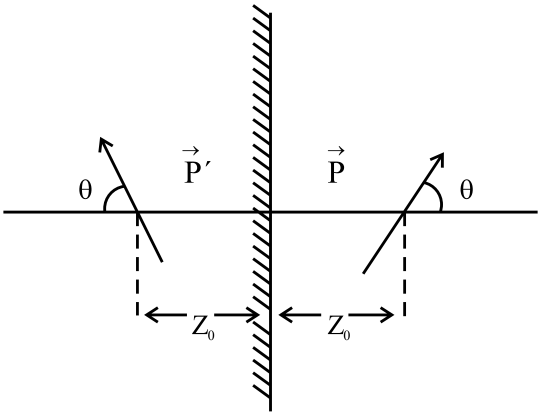


Fig.1.53

QNO 14. Two similar charges are placed at a distance 2d apart. Find appronimately, The minimum radius of a grounded conducting sphere placed midway between them that would neutralize their mutual repulsion.

Solution : The electric field outside the sphere conresponds to the resultant electrical field of the two given charges + q and two image charges $+q'$. By the method electrical images. $q' = {q\,a}/{d}$ and they are to placed at the two sides of the centre of the sphere at the same distance $d' = \frac{a^2}{d}$ from it For each charge + q, besides acted by repulsive force of + q, There is also the altraction exerted by the two image charge, For the resulant force to vanish, we must have

$$\frac{q^2}{4d^2} = \frac{q^2\,a/d}{\left(d - \frac{a^2}{d}\right)^2} + \frac{q^2\,a^2/d}{\left(b + \frac{a^2}{d}\right)^2}$$

$$= \frac{2q^2 a}{d^3}\left[1 + 3\left(\frac{a}{d}\right)^4 + 5\left(\frac{a}{d}\right)^8 + ...\right]$$

$$\approx \frac{2q^2 a}{d^3}$$

The value of a (a < b) that satisfies the d bove requirement is given by,

$$a \approx \frac{d}{8}$$

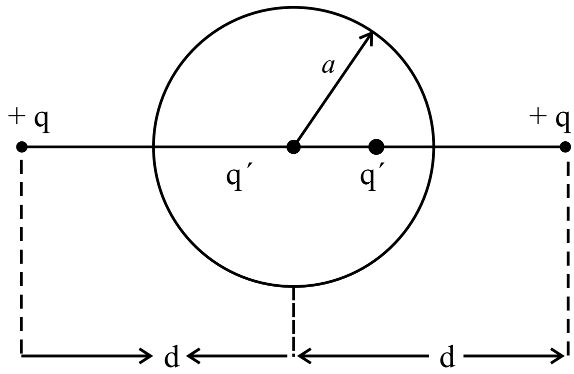


Fig.1.54

QNO 15. Chareges + q at points (q, 0 a) and –q at points (–a, 0, a) above a grounded conducting plane at z = 0, Find

(a) The total force on charge + q

(b) The work done against the electrostatic forces in arranging this distribution of charges

(c) The surface charge density at the point (a, 0, 0).

Solution : The method of image charges implies at + q (–a, 0, –a) and –q at (a, 0, –a). The resutlant force exeted on +q at (a, 0, a) by other charges is

$$\vec{F} = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{1}{(2a)^2}\hat{i} - \frac{1}{(2a)^2}\hat{k} + \frac{1}{\left(2\sqrt{2}a\right)^2}\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\right)\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0 a^2}\left[\left(-\frac{1}{4} + \frac{1}{8\sqrt{2}}\right)\hat{i} + \left(-\frac{1}{4} + \frac{1}{8\sqrt{2}}\right)\hat{k}\right]$$

Magintude of the force $|F| = \frac{\left(\sqrt{2}-1\right)}{32\pi\varepsilon_0 a^2} q^2$

Force is acting on × z plane and points to the origin along a direction at angle 45º to the x axis as shown in the figure.

(b) we can build the system by bringing the charges + q and –q from infinity through the path

$$L_1 : z = x, xy = 0$$

$$L_2 : z = -xy = 0$$

symmetrically to the points (a, 0, a) and (–a, 0 a) resectively. when the charges are at (ℓ, o, ℓ) on path L_1 and ($-\ell$, 0, ℓ) on Path L_2 respetively, each of the charges suffers a force $\frac{(\sqrt{2}-1)q^2}{32\pi\varepsilon_0 l^2}$ whose direction is parallel to the direction of the path. so that total work done by the exernal forces is

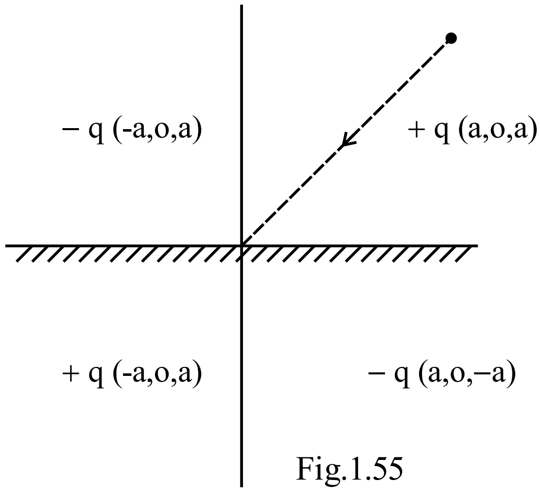


Fig.1.55

$$W - \int_{-\infty}^{a} F dl = 2\int_{a}^{\infty} \frac{(\sqrt{2}-1)q^2}{32\pi\varepsilon_0 l^2} dl$$

$$= \frac{(\sqrt{2}-1)q^2}{16\pi\varepsilon_0 a}$$

(c) Now take case of electric field at a point (a, 0, 0^+) just above the conducting plane; The resultant electric field intensity $\vec{E}_1$ produced by + q at (a, 0, a) and –q at (a, 0 –a) is

$$\vec{E}_1 = \frac{2q}{4\pi\varepsilon_0 a^2}\hat{k}$$

The resultant field $\vec{E}_2$ produced by – q at (–a, 0, a) and + q at (–q, 0, a) is

$$\vec{E}_2 = \frac{2q}{4\pi\varepsilon_0 a^2}\frac{1}{5\sqrt{5}}\hat{k}$$

Hence the total field at (a, 0, 0^+) is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{q}{2\pi\varepsilon_0 a^2}\left(\frac{1}{5\sqrt{5}} - 1\right)\hat{k}$$

So the surface charge density is

$$\sigma = \varepsilon_0 E = \frac{q}{2\pi\varepsilon_0 a^2}\left(\frac{1}{5\sqrt{5}} - 1\right)$$

UNIT 2 : Dielectric Properties of Matter

1.1 Objective

In this unit you will be acquainted with microscopic as well as macroscopic properties of dielectric. Following topics will be convered :

1. Difference between polar and non polar dielectric

2. Explanation of polarisation and quantitative analys's of bound charges due to polarisation.

3. Idea of electric displacement vector and derivation of Gauss's Law in presence of dielectric.

4. boundary condition at the interface of two different dietectric medium.

5. To find the electric field in different structures/shapes of dielectric.

6. Moleular polarisation and its relation with dielectric constant.

7. Properties of differant types of dielectric.

2.2 Introduction

In electromagnetism, a dielectric is an insulator that can be polarised by an applied. electric field. In our earlier study on electrostatic, we are acquainted with external featuring properties of them. The electric field become lessened, with introduction of dietectric media in place of vaccum, even the dietectic inside them. In this unit we will study The transformational properties of dielectric in presence of electric field.

Basically. There are four mechanism of polarisation :

(a) Electronic or atomic, polarisation

This involves the displacement of the centre of the electron cloud around an atom with respect to the centre of its nucleus under the influence of electric field.

(b) Ionic Polarisation

The ionic polarzation occurs, when atoms formmolecules and is mainly due to a relative displacement of the atomic components of The molecules due to the influenece of electric field.

(c) Dipolar ar Orientation polarsation.

This is due to orientation of the molecular dipoles in the direction of the field, which would otherwise to be distributed randomly due thermal agitation.

(d) Interface or space charge polarisation

This involves limited movement of charges resulting in alligenent of charged dipoles under the electric field. It is usually observed at the grain boundaries or any other iterfae such as electrode material interface.

Also we will study molecular level changes due to electric field, This changes are called polarization. Behaviour of bound charges along with the modified Gauss's law wil be explained in dietectric.

2.3 Classification Dielectric Material

In our everyday experience, most of the materical, that we come into contact can be classified into two distinct branch–these are conductor and dielectric. There are many free electrons in conductor which are not attached to the atoms. They move freely every-where at random. Most of the metals have these properties and each atom has one on two such free electrons. There no free electrons in dielectric each electron is all attached to the atom/

molecule. But the electrons can move slightly within the atom, as a consequence, negative and positive charges get slightly displaced. But in certain dielece trics centre of positive and negative charges of the atom donot coincide with the same point, and they tend to behave as electric dipole. Dielectric material can be classified into distinet catagories one is polar and the other is non-polar.

(a) Polar dielectric :

Dietetrics, in which each atom/molecule has permanent dipole moment even in absence of electric field is called polar dielectric. Let us illustrate an example of dipole moment behaviour of HCl,- Hydrogen and cholrine atom have one and seventeen electrons in their outer orbit, respectively. Their chage districbution is such that their centre of positive and negative charges concide with a single point location, but when they coalasce to form a HCL molecue, the one electron of Hydrogen atom goes to The surrouneling chlorine atom. So Hydrogen becomes positively charged and chlorine atom becomes negatively charged of HCL molecule. The molecule HCL is transformed into an electric dipole. Examples of dipolar molecules are water (H_2O), Ammonia (NH_3), Carbondisulphide (CS_2) and Hydrogen sulphide etc. Dipole moment of different molecule is in the range of (1 – 20) coulmb/meter.

(b) Non polar Dielectric : whose atom/molecule of Dielectric material does not have permanent dipole, is called non polar dielectric. In spite of having no permanent dipole, for non polar dielectric, atoms/molecules of the dielectric, or the dielectric as a whole, can be trans formed to have dipole moment under the influence of external electrical field. Examples of nonpolar molecules are H_2, N_2, CO_2, CCl_4, etc.

2.4 Polarisation

When a dielectric material placed inside electrical field, then each atom/molecute becomes converted to dipole. In additon to this, if the material is polar in character, then atoms molecules become more polarised. Average dipole moment generated under the influence of the electric field $\vec{E}$, alligned along the field, is termed as molecular polarsation, and denoted by the symbol $\vec{p}$ If n is the number of molecule/atom, per unit volume, then,

$$\vec{P} = n\vec{p} \;......(2.4.1)$$

$\vec{p}$ is defined as polaraisation sation per unit volume of the dielectric. Its unit is c/m^2.

Let us take an infinetisimal parallelopiped of dielctric of length 'ℓ' and cross-section δs placed in an electric field $\vec{E}$ alligned along the length (Fig. 2.1). Induced surface density charge σ'_b and $-\sigma_b$ will appear on the plane perpendicular to the direction of the electric field which is due polarsiation. Positive charges will mutually be neutralized by the negative charges inside the dielectric material. Total charges $+\delta_b\delta S$ and $-\delta_b\delta S$ will appear at the two terminal end surface of the dielectric material. So the total dipolemoment polarisation of parallelopiped will be $\delta_b\delta Sl.$ Again the volume of the dielectric is $\delta Sl,$ and its dipole moment is $\vec{P}\delta Sl.$ So,

$$\delta_b\delta S.l = P\delta S.l.$$

$$\Rightarrow P = \sigma_b \text{ ......}(2.4.2)$$

So induced surface charge density on the surface perpendicular to the direction of the electric field is equal to the value of polarisation vector. Even if the plane surface is not perpendicular to the electric field, it can be shown that induced surface density of charge will be $\vec{P}.\vec{n} = \sigma_b$ ......$(2.4.3)$

Where $\vec{n}$ is the unit vector perper clicular to the surface.

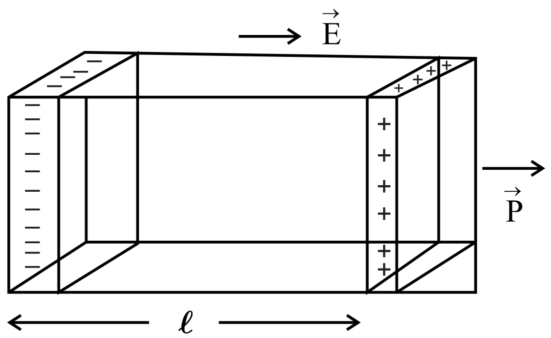


Fig. 2.1

2.4.1 Electrical field due to polarised dielectric

When applied electric field causes polarisation in dielectric, dipole moment is developed in the dielectric. So these dipoles will certainly create electric field on its own. Let us try to find the potential due to this dipoles.

Potential due to a single dipole $\vec{p}$ is given by

$$V\left(\vec{r''}\right) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{P}\cdot\hat{r}''}{r^{2''}} \text{ ... ...}(2.4.4)$$

Where $\vec{r''}$ is the vector from the dipole to the point at which we are finding the potential

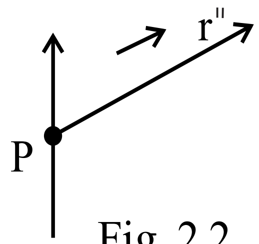


Fig. 2.2

(Fig. 2.2) Now the dipole moment $\vec{P} = \vec{p}dv'$ in each volume element $d\tau'$, so the potential

is $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\vec{P}(\vec{r}\,').\hat{r}\,"}{\vec{r}\,"} d\tau \ldots\ldots(2.4.5)$

But we know from vector,

$$\nabla'\left(\frac{1}{\vec{r}\,"}\right) = \frac{\hat{r}\,"}{r^{2}\,"}$$

where the differentiation is with respect to the source co-ordinates (r'), so we have,

$$V = \frac{1}{4\pi\varepsilon_0}\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\vec{r}\,"}\right) d\tau'$$

On integrating by part's

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_v \vec{\nabla}'.\left(\frac{\vec{P}}{r"}\right)d\tau' - \int_v \frac{1}{\vec{r}\,"}(\vec{\nabla}'\vec{P})d\tau'\right]$$

using Gauss's divegence theorem

$$V = \frac{1}{4\pi\varepsilon_0}\oint_\tau \frac{1}{\vec{r}''}\vec{P}.\vec{ds} - \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r"}\left(\vec{\nabla}\cdot\vec{P}\right)d\tau' \ldots\ldots(2.4.6)$$

The first term of integration, we get potential due to surface charge

$$\sigma_b = \vec{P}\cdot\hat{n} \ldots\ldots (2.4.8)$$

Where $\hat{n}$ is the unit normal vector to the surface. The second term of the intgrand will give the potential of a volume charge

$\rho_b = -\vec{\nabla}\cdot\vec{P}$ ... ...(2.4.9)

Using the above defined term, equation (2.4.6) can be written as,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{\vec{r}''} ds' + \int \frac{1}{4}\frac{1}{4\pi\varepsilon_0}\int \frac{\rho_b}{\vec{r}''} d\tau' \text{ ... ...(2.4.10)}$$

The experssions for V contains two terms, first is $\sigma_b = \vec{P}\cdot\hat{n}$ which is the due to induced surface charge. Second term is from the distribution of charge throughout the volume of the dielectric, $\rho_b = -\vec{\nabla}\cdot\vec{P}$, produced due volume density of charge. In our earlier discussion we have assumed uniform distribution volume charge density, i.e. $\vec{\nabla}\cdot\vec{P} = O$, as a result, excluding the outer surface charge density of the dietetric, every- where in the volue + ρ and –ρ volume density of charges neutrilizes each other.

2.4.2 Bound Charges

We have observed that polarisation causes the appearance of surface charge on the outer surfaces and charges inside the volume of the dielectric. These charges are bound or fixed with the individual atom or molecule, and cannot move freely. So no current is generated due to this. So they are callted bound charges. The character of free charges in a conduction is different from the bound charges of dielectric as they can move freely to conduct electricity on with drawal of the external field, bound charges are removed due to reunification of positive and negative charges.

2.5 Gauss's Law in Dielctrics

Electrostatic field in the dielectric material is modified due to polarization and not like as in vaccum. In the Fig. 2.3 (a) a parallel plate capacitor, of metallic plate, each of area S, medium is vacuum inside them. In the Fig. 2.3 (b) An identical capacitor with a dielectric inside them.

Both the plates of each capacitors are charged with +q and –q, respectively. A gaussian surface abcd encirculing the +q charges have been drawn. whose ab and cd surfaces are parallel to the plates of the capacitor, other planes are perpendicular to the plate. When the medium is vacuum as in Fig 2.3(a) and the electrical field intensity is $\vec{E}_0$, then applying Gauss's Theorem in abcd plane, we get, S being their area of plate,

$$\int \vec{E}_0.\vec{ds} = q/\varepsilon_0$$

$$E_0 \cdot S = q/\varepsilon_0 \quad \ldots\ldots(2.5.1)$$

Observe that Gaussian integral is non zero only for cd plane.

Induced charge will be developed due to polarisation adjacent to the plates of the capacitor and two terminal end of the dielectric, as shown in Fig. 2.3 (b) Let $-q'$ be The induced bound charges near the positive plate of the capactior. $\vec{E}$ is the electrical field intensity inside the dielectric. Applying Gauss's Theorem in abcd plane, as before, we get

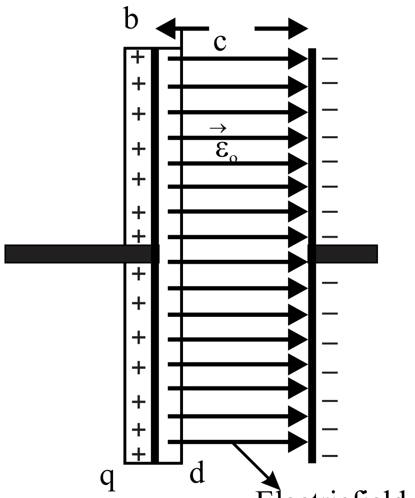


Fig. 2.3a

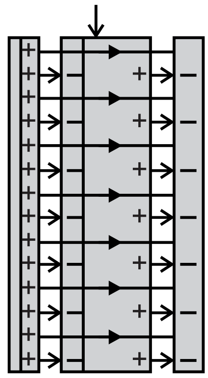


Fig. 2.3b

$$\oint \vec{E} \cdot \vec{ds} = E.S = (q - q')/\varepsilon_0 \quad \ldots\ldots(2.5.2)$$

From equation (2.5.1) and (2.5.2) we get

$$E = \frac{q}{\varepsilon_0 S} - \frac{q'}{\varepsilon_a S}$$

$$E = E_O - \frac{q'}{\varepsilon_0 S} \ldots\ldots(2.5.3)$$

If V_0 and V is the potertial difference between two plates of capactior with and without dielectirc, then $\frac{E_0}{E} = \frac{V_0}{V} = k$ where k is dielectric constant As $E < E_0$, k is greater than 1.

From equation (2.5.3) we get

$$\frac{q}{k\varepsilon_0 S}=\frac{q}{\varepsilon_0 S}-\frac{q'}{\varepsilon_0 S}$$

Or, $q' = q(1-1/k) \ldots\ldots(2.5.4)$

From equation (2.5.4) it is clear That $q' < q$

Now equation (2.5.2) can be written in the following way

$$\varepsilon_0 \oint \vec{E}\cdot\vec{ds} = q - q' = \frac{q}{k}$$

Or, $\varepsilon_0 \oint k\cdot\vec{E}.\vec{ds} = q \ldots\ldots(2.5.5)$

Equation (2.5.5) is more advantageous form of Gauss's law to apply in dietectric. As the quantity k has been included in intagral of the left side, there is no need to include induced charges on the right hand side of the integral. $\varepsilon_0 k$ is the permittivity of of the dielectric medium and denoted by $\varepsilon.$ In our earlier discussion. $-q'$ and q' have been mentioned to be the charges residing on the end surfaces of diectric, Their surface density of charges will be $-\sigma_b = -P$ and $\sigma_b = P,$ respectively.

From equation (2.5.3) we get,

$$E = kE - P/\varepsilon_0$$

Or, $\varepsilon_0 KE = \varepsilon_0 E + P \ldots\ldots(2.5.6)$

Here E and P are all vector quantities, Equation (2.5.6) can be written, in the following way $\varepsilon_0 K\vec{E} = \varepsilon_0 \vec{E} + \vec{P} \ldots\ldots(2.5.7)$

Expression on the right hand side of the equation (2.5.7) is termed as Maxwell electrical displacement and denoted by $\vec{D}$

From equation (2.5.5) we get

$$\oint \vec{D}.\vec{ds} = q$$

Again we know $q = \oint \rho_f dv,$ where ρ_f is free volume charge density, and integral has been covered all over the gaussian surfaces. Applying divergence theorem, we can show that

$\vec{\nabla}\cdot\vec{D} = \rho_f$ ... ...(2.5.8)

Equation (2.5.8) is the differential form of the Gauss's law in dielectric medium.

Equation (2.5.7) can be written in the following way,

$\vec{P} = \varepsilon_0 (k-1)\vec{E} = \varepsilon_0 \chi \vec{E}$ ... ...(2.5.9)

As k > 1, $\chi = k-1$ is positive, and denoted as electrical susceptibility.

2.5.1 Relation between $\vec{E}, \vec{P}$ and $\vec{D}$

The equation (2.5.7) written as $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$, then we can get a clear relation between $\vec{D}, \vec{E}$ *and* $\vec{P}$. Though $\vec{D}$ depends only on free charges, $\vec{E}$ and $\vec{P}$ vectors depend both on free and bound charges. In linear isotropic dielectric, $\vec{D}, \vec{E}$ and polarization vector are parallel To each other, i.e. $\vec{D} = K\vec{E}$, where k is a scalar quantity In nonlinear, anisotropic dietectric $\vec{D}, \vec{E}$ and $\vec{P}$ are not parallel. Here K is represented by tensor quantity.

2.6 Boundary Condition in Dielectric Medium

(a) Let AB be boundary between two dietectric media, which is homogenous and isotropic. Consider The interface between 1 and 2, and imagine small pill box shaped Gaussian surface intersecting the interface [Fig. 2.4 (a)]. Its height and the area covered by the curved surface is very small. Let ΔS be area cut out by the pill box on the interface. If σ be the surface chage density of free charge on the interface, The application of Gauss's law to The pillbox yields–

$\vec{D}_2 \cdot \hat{n}\Delta S - \vec{D}_1 \cdot \hat{n}\Delta S = \sigma \Delta S$ ... ...(2.6.1)

Where $\hat{n}$ is the unit vector pointing from medium 1 to medium 2. Flux over the curved surface is negligible and does not contribute to the equation (2.6.1) Neglecting negligible volume of the pill box, we get, $\left(\vec{D}_2 - \vec{D}_1\right)\cdot\hat{n} = \sigma$ ... ...(2.6.2)

Or, $D_{2n} - D_{in} = \sigma$ ... ...(2.6.3)

It is clear from equation (2.6.2) or (2.6.3), that discontinuity in the normal component of the electric displacement in moving from one medim to other medium is given by the surface density of free charge on the interface between the media. Normal compant of $\vec{D}$ is conlinous across it when there is no free charge at the interfaec.

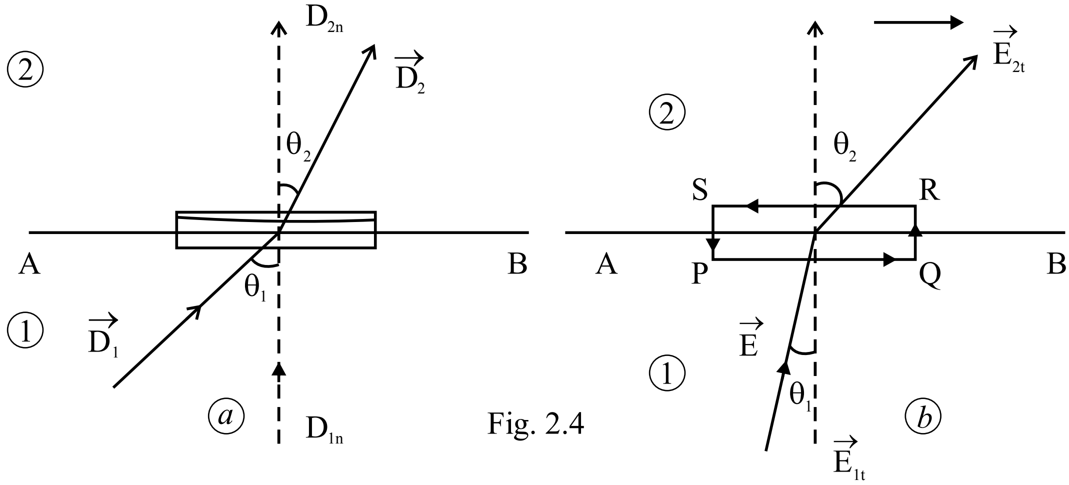


Fig. 2.4

(b) Let AB be The boundary between two dielectric media 1 and 2. Take a closed path PQRS across The boundary AB [Fig. 2.4 (b)] Its height QR and SP are very small and neghigible and the length PQ = RS = dl. Let $\vec{E}_1$ and $\vec{E}_2$ are electric field vectors in media 1 and 2 at an inclination θ_1 and θ_2 with the norma to the boundary The work done in moving an unit positive charge around the path PQRSP is zero $\oint \vec{E}.\vec{dl} = O$

$$\oint E \sin\theta \, dl = 0$$

$$E_1 \sin\theta_1 \cdot dl + \{-E_2 \sin\theta_2 \cdot dl\} = 0$$

$$E_1 \sin\theta_1 - E_2 \sin\theta_2 = 0 \text{ ......}(2.6.4)$$

But $E_1 \sin\theta_1 = E_{1t}$ and $E_2 \sin\theta_2 = E_{2t}$

E_{1t} and E_{2t} are The tangertial component of electric fields on both sides dielectric boundary, So, $E_{1t} = E_{2t}$ ... ... (2.6.5)

Thus the tangetial component of The electric field is continuons across The interface between two media.

(c) Refraction of Eelectical lines of Force :

we know, for charge free interface between two dielectric medium

$D_{1n} = D_{2n}$

Or, $D_1 \cos\theta_1 = D_2 \cos\theta_2$

Or, $\varepsilon_0 K_1 E_1 \cos\theta_1 = \varepsilon_0 K_2 E_2 \cos\theta_2$

Or, $K_1 E_1 \cos\theta_1 = K_2 E_2 \cos\theta_2$ ... ...(2.6.6)

and $E_1 \sin\theta_1 = E_2 \sin\theta_2$ ... ...(2.6.7)

So we get from equations (2.6.6) and (2.6.7)

$K_1 \cot\theta_1 = K_2 \cot\theta_2$ ... ...(2.6.8)

This is the law of refraction for electrical lines of force when $K_2 > K_1$

Then $\cot\theta_1 > \cot\theta_2$ or, $\theta_2 > \theta_1$ implies that when the dielectric constant of medium two is greater than medium one, electrical lines of force in medium two, will move away from the normal at the interface. This is in contrast opposite to the normal refraction of light rays.

2.7 Energy Density within Dielectric Medium

Consider a system free charges embedded in a dielectric medium. The inerease in the total energy when a small amount of free charge $\delta\rho_f$ is added to the system is given by

$$\delta W = \int \phi(\vec{r})\delta\rho_f dV \text{ ... ...(2.7.1)}$$

Where the integral is taken over all space, and $\Phi(\vec{r})$ is the electrical potential. Here we have assumed that the original charges and the dielectric and held fixed, so that no mechanical work is done. From equation (2.7.1) we get, $(\vec{\nabla}\cdot\vec{D} = \rho_f)$

$$\delta W = \int \Phi\vec{\nabla}\cdot\delta\vec{D}\ dV \text{ ... ...(2.7.2)}$$

Where $\delta\vec{D}$ is the charge in electric displacement due to increase in charge.

Using the vector identity,

$$\vec{\nabla}.\left(\Phi\vec{D}\right)=\vec{\nabla}\Phi.\vec{D}+\Phi\left(\vec{\nabla}\cdot\vec{D}\right)$$

We get

$$\delta W=\int\vec{\nabla}\cdot\left(\Phi\delta\vec{D}\right)d\nu-\int\vec{\nabla}\Phi\cdot\delta\vec{D}\,d\nu\ \ldots\ldots(2.7.3)$$

giving, $\delta W=\int\Phi\delta\vec{D}\cdot\vec{ds}-\int\vec{\nabla}\Phi\cdot\delta\vec{D}\,d\nu\ \ldots\ldots(2.7.4)$

If the dielectric medium is of finite spatial extent, then we can neglect the surface term to give,

$$\delta W=-\int\vec{\nabla}\Phi\cdot\delta\vec{P}d\nu=\int\vec{E}\cdot\,\delta\vec{P}d\nu\ \ldots\ldots(2.7.5)$$

Assuming $\vec{D}=\varepsilon_0 K\vec{E}$ where K is the dielectric constant, the change in energy associated while $\vec{D}$ has been increased from 0 to $\vec{D}(r)$ at all points in space is given by

$$W=\int_0^D\delta W=\int_0^D\int\vec{E}\cdot\delta\vec{P}d\nu\ \ldots\ldots(2.7.6)$$

Or, $W=\int\int_0^E\frac{\varepsilon_0 K\delta\left(E^2\right)}{2}d\nu=\frac{1}{2}\int\varepsilon_0 KE^2d\nu$

Which rediuces to

$$W=\frac{1}{2}\int\vec{E}\cdot\vec{D}\,d\nu\ \ldots\ldots(2.7.7)$$

So the electrostatic energy density inside a dietectric medium is given by

$$U=\frac{1}{2}\vec{E}\cdot\vec{D}\ \ldots\ldots\ (2.7.8)$$

2.7.1 Potential Energy of Dipole in Electrical Field

When a dipole placed in an electric field, two equal and opposite forces $\vec{F}$ and $-\vec{F}$ on the charges q and –q, which constritules a couple (Fig. 2.5).

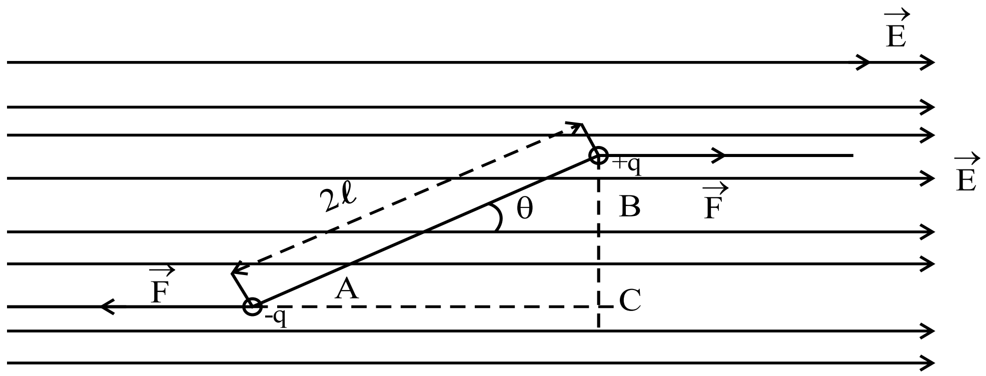


Fig. 2.5

The moment of the couple or toque = Force × perpendicular distance

But F = qE and from the ΔABC, $BC = AB\sin\theta$

$\therefore \tau = qE2l\sin\theta$ [as AB = 2l = length of the dipole]

Now, p = 2lq = dipole moment

$\tau = PE\sin\theta$

which forms a vector, $\vec{\tau} = \vec{p} \times \vec{E}$ ......(2.7.9)

This dipole will be rotated by the couple τ in the direction of the field. Let dw be the work done in rotating the dipole Through a angle $d\theta$,

$dw = \tau.d\theta$

Total work done w in rolating the dipole from angle θ_1 to θ_2. is $w = \int_{\theta_1}^{\theta_2} \tau \cdot d\theta$

The work done is stored in the dipole as potential energy U

$\therefore U = W - pE(\cos\theta_2 - \cos\theta_1)$

Or, $U = -pE[\cos\theta_2 - \cos\theta_1]$

If the initial and final positions are $\theta_1 = 90^\circ$ and $\theta_2 = \theta$ Then,

$U = -pE\cos\theta$

Or, $U = -\vec{p}\cdot\vec{E}$ ... ...(2.7.10)

2.8 Electronic Polarisation

Consider an atom in an electric field of intensity '$\vec{E}$', since the nucleus of charge +ze and surronding encircling electron cloud of charge '–ze' of the atom have opposite charges

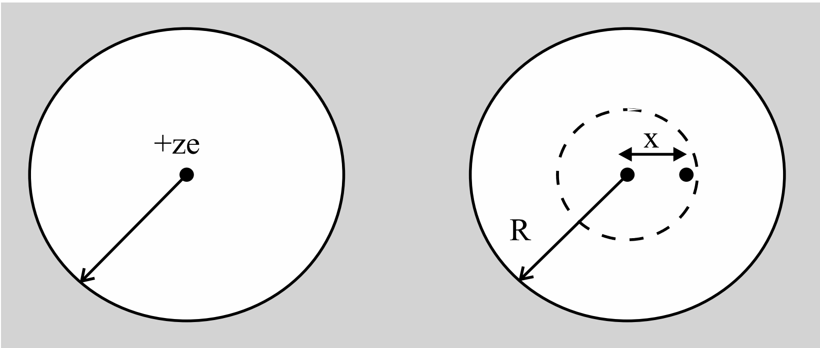


Fig. 2.6

and acted upon by Lorentz force. As a consequence, nucleus moves in the direction of The field and electron cloud in the opposite direction. As electron cloud and nucleus gets displaced from their normal equillibrium positions, an allractive force between them is built and the separation continues until coulomb force F_C is balanced by The Lorentz force F_2, until a new equillibrium state is ereated.

Let ρ be the charge density of the sphere.

$$\rho = \frac{-ze}{\frac{4\pi}{3}R^3}$$

where '–ze' is the total charges in the sphere. So the negative charge in the sphere of radius x, $q_x = \rho\frac{4}{3}\pi x^3$

$$= \frac{-ze}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi x^3\right)$$

$$q_x = \frac{-ze}{R^3}x^3 \text{ ... ...(2.8.1)}$$

Allractive coulomb force between nucleus and electrons

$$F_C = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_x q_p}{x^2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0 x^2}\left(\frac{-zex^3}{R^3}\right)ze$$

$$F_C = \frac{-z^2e^2x}{4\pi\varepsilon_0 R^3} \;\ldots\ldots(2.8.2)$$

Force experienced by displaced, nucleus in electric field internsity $\vec{E}$ is

$$\vec{F}_L = \vec{E}_{q_p} = Ze\cdot\vec{E} \;\ldots\ldots(2.8.3)$$

Since, $F_L = F_C$, we can write

$$\frac{-z^2e^2x}{4\pi\varepsilon_0 R^3} = zeE \;\ldots\ldots(2.8.4)$$

Or, $E = -\dfrac{zex}{4\pi\varepsilon_0 R^3} \;\ldots\ldots(2.8.5)$

Again we know $E = \dfrac{dipole\,moment}{\alpha_e}$

where α_e is represented as electronic polarizability.

From equation (2.8.5), we get

$$\frac{-zex}{4\pi\varepsilon_0 R^3} = \frac{-zex}{\alpha_e}$$

So, $\alpha_e = 4\pi\varepsilon_0 R^3 \;\ldots\ldots(2.8.6)$

Hence electronic polarisibility is directly proportinal to the radius of the atom.

2.9 Electrical Field inside a cavity in the Dielectric

Spherical Cavity : Let us imagine a speherical cavity inside the dielectric whose centre at O and radius 'r'. The size of the cavity is small compared to the dielectric material, but large enough compared to size of the molecule. The electric field inside The dielectric is E. Electrical force on a unit positive charge placed at O is E where $E' = E + E_P$

E_P is electrical intensity due to the induced charge on the surface of the cavity. We use spherical co-ordinate system to find E_P. If $\vec{P}$ is the polarisation vector, then surface charge density on the surface of the cavity $\sigma = \vec{P} \cdot \hat{n},$ where $\hat{n}$ is the unit vector perpendicur to the surface so, total surface charge on an elementray area ds is

$$dq = \sigma ds = \vec{P} \cdot \hat{n} ds = P ds \cos\theta$$

Electrical field, intensity due to this charges at 'O'

$$dE_P' = \frac{1}{4\pi\varepsilon} \frac{P ds \cos\theta}{r^2}$$

We will take the component which is parallel to $\vec{E}$, Horizontal component is

$$dE_P = dE'_P \cos\theta = \frac{P ds \cos^2\theta}{4\pi\varepsilon_0 r^2}$$

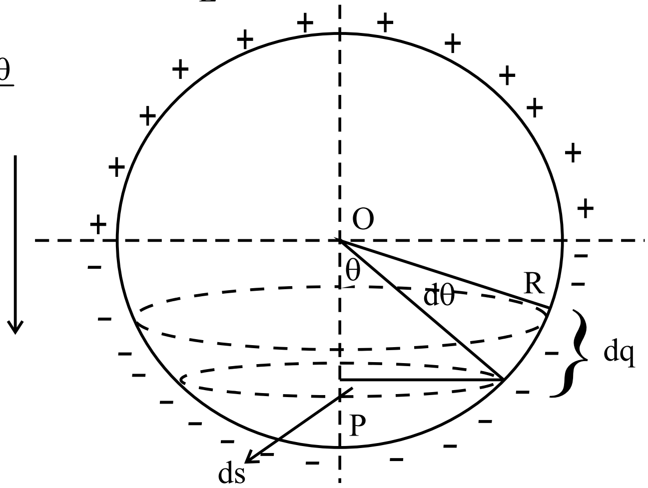


Fig. 2.7

Now elemental surface area between θ and $\theta + d\theta$, is

$$ds = 2\pi r^2 \sin\theta d\theta$$

So,

$$dE_P = P \cdot 2\pi r^2 \sin\theta \cos^2\theta d\theta$$

Or, $dE_P = \frac{P}{2\varepsilon_0} \sin\theta \cos^2\theta d\theta$

Intequrating $\theta = 0 \; to \; \theta = \pi,$ we get

$$E_P = \frac{p}{2\varepsilon_0} \int_0^{\pi} \sin\theta \cos^2\theta d\theta = \frac{P}{2\varepsilon_0} \cdot \frac{2}{3} = \frac{P}{3\varepsilon_0} \quad \ldots\ldots(2.9.1)$$

So the total intensity at the centre of the cavity $\vec{E}' = \vec{E} + E_P = \vec{E} + \frac{\vec{P}}{3\varepsilon_0}$ ... ...(2.9.2)

If the dielectric material is kept inside a parallel plate capacitor, Then electrical field intensity at the centre of the cavity is $\vec{E}_m = \vec{E}_0 + \vec{E}_i + \vec{E}_P$ ... ...(2.9.3)

where $\vec{E}_0$ is the itensity due to the chargeson the capacitor plates. This field induces potarization inside the dietetric, and induced surface charge density + P and –P on the terminal surface of the dielectric $\vec{E}_i = -P/\varepsilon_0$

Electrical field inside the dielectric $\vec{E} = \vec{E}_0 + \vec{E}_i = \vec{E}_0 - \vec{P}/\varepsilon_0$

So the electrical field intensity at the centre of the cavity is less than $\vec{E}_0$, but it is greater than the field inside the dielectric.

2.9.1 Atomic and Molecular Polarisation : Clausious–Mossotti selation

Now we will, find out the relation between relative permillivity and molecular/atomic polarisation. Let us now explore the field intensity at the centre of sphere of radius 'r' inside the dielectric material. All the molecules inside the sphere gets polarised along with the polarisation entire dielectric. But all the dipoles inside it contributing to the field at the centre gets neulralised or mitigated due to the vector sum of the evenly distributed dipoles (fields). So $\vec{E}_m$ is effective intensity of a molecule kept at the centre 'O'.

We know $\vec{D} = K\varepsilon_0\vec{E} = \varepsilon_0\vec{E} + \vec{P}$

where $\vec{E}$ is The electrical intensity inside the dielectric.

Or, $\vec{P} = (k-1)\varepsilon_0\vec{E}$

From $\vec{E}_m = \vec{E} + \vec{E}_P$

$$= \vec{E} + \frac{(k-1)}{3}\vec{E} = \frac{k+2}{3}\vec{E}$$

Or, $\vec{E} = \frac{3}{k+2}\vec{E}_m$

So, $\vec{P} = (k-1)\varepsilon_0 \times \frac{3}{k+2}\vec{E}_m$ ... ...(2.9.4)

Or, $\vec{P} = \frac{3(k-1)}{(k+2)} \varepsilon_0 \vec{E}_m$ ... ...(2.9.5)

If P_m is the dipole moment of molecule generated due the electric field intensity E_m, so the dipole moment per unit field strength, so we say molecular polarisibility $\alpha = \frac{P_m}{E_m}$

If 'n' is the numbe of molecule per unit volume, Treating $\vec{P}_m$ as vector

$\vec{P} = n\vec{P}_m = \alpha n \vec{E}_m$

Hence, $\alpha = \frac{3\varepsilon_0}{n} \frac{(k-1)}{k+2}$ ... ...(2.9.6)

Above relation of equation (2.9.6) is known as **clausius-Mossotti relation.** Physical implication of this relation is that we can get the entrie macroscopic propertics i.e. we can get the value of molecular polarsability from the relative permillivity. Thus, from a measurement of k, it is possible to get important quautitative information about molecular structures.

From electronic polarisation, we know that molecular polarisibility from equation (2.8.6) is $\alpha = 4\pi\varepsilon_0 a^3$

Using classius-Mossotti relation, we get

$\frac{k-1}{k+2} = \frac{n\alpha}{3\varepsilon_0} = \frac{n 4\pi\varepsilon_0}{3\varepsilon_0} = n \cdot \frac{4}{3}\pi a^3 = V$ ... ...(2.9.7)

where V is the volume of total one unit volume of molecules.

From equation (2.9.7), we have

$a^3 = \frac{3}{4 \times n} \cdot \frac{k-1}{k+2}$ Or, $a = \left[\frac{3}{4 \times n} \cdot \frac{k-1}{k+2}\right]^{1/3}$ ... ...(2.9.8)

Equation (2.9.8) gives the relation between atomic radius and dielectric constant (k).

2.10 Polar Dielectrices and The Langevin Debye Formula

Molecules like CH_3CL, H_2O, HCl, cthyl acetate carries electric dipole moment even in the absence of electric field. However, The net dipole moment is negligiby small since all the dipoles under continous thermal agitation, are oriented randomly when there is no external electric field. In the presence externally applied field, individual dipoles experience torques, which tend to allign them along the field direction. As a result the net dipole moment becoems large.

Polarisability has been calculated based on the principle of statistical Thermodynamics. Under this principle, in thermal eqnillibrium, the probability of finding a molecule with potential energy U is proportional to e^{-U/K_BT}, where k_B is The Boltzman constat and T is the absolute temperature. The potential energy of a dipole moment $\vec{P}$ in an electric field is, $V = -\vec{P}\cdot\vec{E} = -PE\cos\theta$ ... ...(2.10.1)

Assuming, local field is solely to be the electric field, the probability that a dipole will have orientation θ with respect to the field is $e^{PE\cos\theta/K_BT}$. If < P > is The average polarisibility of dipolar molecule, at a particular temperature is given by the Langevin formula

$$\frac{<P>}{P} = \coth\alpha - \frac{1}{\alpha} \text{ ... ...(2.10.2)}$$

where $\alpha = \dfrac{PE}{K_BT}$

The fig. (2.8) shows The variation of $\dfrac{<P>}{P}$ as a function of α. At large electrical field

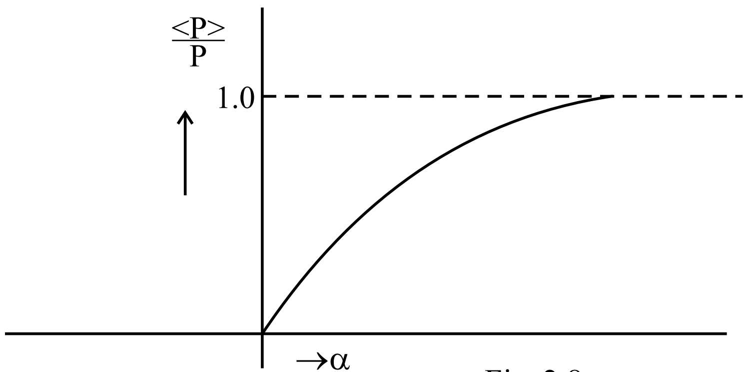


Fig. 2.8

strengths or at low temperatares i.e. where $\alpha = \dfrac{PE}{K_BT} >> 1$, Lengevin predicts $\dfrac{<P>}{P} = 1$. which states that nearly all the polar molecules have been alligned with the electric field. i.e. almost saturation $\dfrac{<P>}{P}$

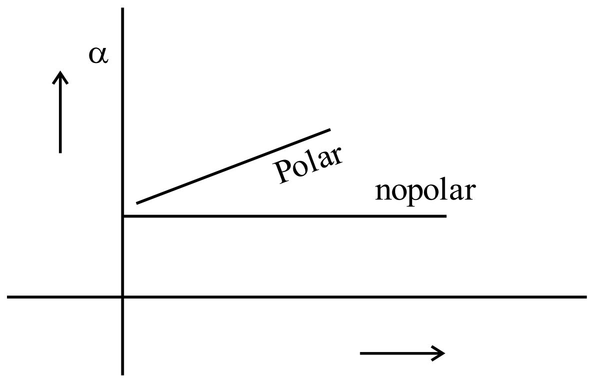
while for small values of $(\alpha << 1)$, i.e. normal fields and higher temperatures equation (2.10.2) reduces to $\frac{<P>}{P} \approx \frac{\alpha}{3}$ Or, $<P> = \frac{P^2 E}{3K_B T}$ ... ...(2.10.3)

which indicates a linear relationship between < P > and E. Thus a polar dielectric is normally linear. Now the polarsibility α is defined as the molecular dipole moment per unit field

$$\alpha \approx \frac{<P>}{E} = \frac{P^2}{3KT} \quad \ldots \ldots (2.10.4)$$

Equation (2.10.4) shows the temperature dependence of polarisibility. This equation holds pretty well for small values of P and E and for large enongh T which we can presume as normal conditions.

The total polarization for dilute gas can be written as



$$\alpha = \left[\alpha_e + \alpha_i + \frac{P^2}{3KT}\right] \quad \ldots \ldots (2.10.5)$$

where α_e and α_i are electronic and ionic polarisibility, respectively.

Equation (2.10.5) is known as **Langevin–Debye equation.**

From clausius–Mossolti equation and equation (2.10.5), we get,

$$\frac{k-1}{k+2} = \frac{n}{3\varepsilon_0}\left(\alpha_e + \alpha_i + \frac{P^2}{3KT}\right) \quad \ldots \ldots (2.10.6)$$

This equation is known as Debye equation. From this equation we can find the value of dipole moments and polarization from measurement of gases. For polar molecules $\alpha vs\, 1/T$ will be a straight line. But equation (2.9.6) shows that for nonpolar molecules α versus $1/T$ graph will be a straight line parallel to 1/T – axis (Fig. 2.9) The intercept of the straight line for polar molecules gives the value of $(\alpha_e + \alpha_i)$ and the slope of the line gives P.

The variation of dielectric properties with the frequency of an applied ac field also interesting. Due to interia of heavy polar molecules they cannot follow the rapid change in the direction of the applied ac field. For this at higher frequencies (in the micro wave region of above) the polar contribution to the dielectric constant begins to fall with frequency. But because of smaller incrtia of electric the electronic polarisibility remeins almost unchanged upto optical frequences.

2.11 Some special Properties of Dielectric Material

Here we will discuss some specific propertis of dielectric material which is of immense use in engineering and as sensors, etc

1. Ferroelectric Materials :

These are crystalline materials that displays electrical polarisations switehable by an external field. Ferro electric crystals have high dietectric constant and each unit cell of ferroelectric crystals carries reverisble electrical cell.

Ferro electric property depends on temperature and this property vanishes at a certain

critical temperature–dietectric property vauishes rapidly with temperature. Relation between dietectric constant, temperature and critical temperature is given by

$$K = \frac{C}{T - T_e} + K_0$$

Here T_c is the critical temperature, c is constant and K_0 is the contribution to dielectric component from electronic dielectric constant. Examples are Barium Titanate ($BaTiO_3$) sodium nitrate and Rochelle salt.

2. Piezeo Electricity :

The process of creating electrical polarisation by mechanical stress is called piezo electric effect. Contrary to this, inverse piezo electric effect is observed, when electric field is applied–The material gets strained and directry proportional to the strength of the electrial field.

Examples are, quartz crystal, Rochelle saltet. Among The piezo electric semiconductor are Gatts, ZnO and Cds – which are mainly used utrasomic amplifiers, electromic watch, microphone etc.

3. Electret :

This can be considered a piece of diectric materical with the presence of quasi-permanent real charges on the surface or in the bulk of the material or frozen-in-alligned dipole. Some dipole moment remains–even when the electrial field is removed. Some organic paraffin, and some plastic exhibits these propertics.

When These materials are polarised in molten state, and There after solidefied, they retain Their dipolar characteris tics, and a permancent dipole generated. It is called thermal electret. Some materials are called photo electret when are transformed by light and electric field to dipole propertics.

4. Dielectric Break down :

All dietectric material retain their property until high enongh field to destroy their characteristics, allowing large flow of current, this happens due to removal of electrons from the atomic orbit by strong electric field mainly. Also some breakdown is observed by the effect of following agent–intrivnsic, thermal, electrochemical, deffect and discharge, breakdown.

5. Dielectric Relanation Time :

It takes a certain amount of time for a dietectric to be fully polarised when subjected to an electric field. It is observed that the electromic and ionic polarisation is attained instaneonsly, if we cosider high frequencies ($10^7 – 10^{17}$/sec) and not The optical frequencies. Dielectric loss, at these frequencies, is mainly due to relaxation effect of the permanent dipoles. A molecule in dielectric, which tries to allign with the applied electric field, is effected by the opposing forces of adjacent molecule. This is the phenomenon of relaxation. Polarisation of the dielectric, when influenced by an atterating electric field, does not conform to proportionatc transformational gain. Rather a hysterisis is observed in polarisation. It has been observed that the platies of the capacitor gets charged again even after being discharged to meutralise the plates from the first follow up charging. Hence, sometimes a certain amount small current flow has been observed Electrical energy is lost due to hyserisis of the dielectric and flow of current, dielectric material gets heated. Motecules cannot orient harmoniously and swiftly with high frequency alternating field. So There is no loss of energy due to hysteris.

2.12 Summary

1. Dielectrics are insulators that support charge. The dielectric constant K indicates polarisibility of dielectric. Each of the polarisation mechanism has a chargeteristics relaxation time (frequency).

2. Gauss's law for dielectric is relates free charge to the displacement vector. $\vec{E}$ and $\vec{D}$ Follow the boundary condition at the interface of two different dietectric media. Refraction law has been duduced.

3. Microscopic properties have been discussed after studying macroscopic properties. Inter relation between molecular polarisabiity and dielectric constant has been deduced by studying the electric field intensity in microscopic spherical cavity deep inside dielectric substance.

4. Clausius–Mossotti reation has been deduced as $\alpha = \frac{3\varepsilon_0}{n}\frac{k-1}{k+2}$ [see equation 2.9.6)]

Again, Molecular polarisibility $\alpha = 4\pi\varepsilon_0 a^3$ (see quation (2.8.6). combining these two equation, we have establised the relation between atomic radius (a) and dielectri constant (k).

5. We have alo established the expression of molecular potarisibility (α) for polar dielectrics given by $\alpha = \alpha_e + \alpha_i + \frac{P^2}{3KT}$ [see equation (2.10.5)]

which is known as Langevin–Debye equation.

6. α versus 1/T for polar and nonpolar dielectrics have been plotted (see Fig. 2.9) and importance of the graph has been discussed.

2.13 Review Questions and Answers

1. What is polar and non polar dielectric? How They behave in an exeternally applied electric field.

Answer : See article 6.3.

2. Find the electric potential inside a polarised dielectric.

Answer : See article 2.4.

3. Prove The continunity of normal and tangentral component of electrical field intensity at the interface two different media.

Answer : See boundary condition article 2.6.

4. Establish The claussius–Mossotti relation.

Answer : See claussium–Mosotti relation, article 2.9.1.

5. Define orientation polarisation.

Answer : When an electric field is applied in a dielectric medium with polar molecules, the electric field tries to allign these dipoles along its field direction, due to that there is a resultant dipole moment in the dielectric material and this process is called orientation polarisation. $\alpha_P = \dfrac{P^2}{3KT}$

6. Define local or internal or Lorentz field.

Answer : In a dielectric material. The field acting at the location of an atom is called as local field or internal field 'E_i'.

The internal field E_i must be equal to the sum of the applied field and the field due to The location of the atom by the dipoles of all other atoms.

E_i = E + The field due to all other atoms.

7. What is electric polarisation?

Answer : It is defined as production of electric dipoles by the applied field. It is due to the shifting of charges in the elietectric by the applied electric field.

8. Mention The different break down mechanism in dietectric material.

Answer : (i) Intrinsic and avalanche break down.

(ii) Thermal break down

(iii) Chemical and electrochemical break down

(iv) Discharge break down

(v) Defect break down

2.14 Problems and Solutions

1. The dielectric constants of a Helium gas at NTP is 1·0000685. Calculate the electric

polarizability of Helium atoms if the gas contains $2{\cdot}7 \times 10^{26}$ atoms/m^3. Calculate the radius of the Helium atom.

[Given $\varepsilon_0 = 8{\cdot}854 \times 10^{-12} Fm^{-1}$]

Solution :

Relative permitivity $\varepsilon_r = 1{\cdot}0000685$

No of atoms of the Helim gas N = $2{\cdot}7 \times 10^{26}$ atms/m^3

Permitivity of free space $\varepsilon_0 = 8{\cdot}854 \times 10^{-12}\ F/m$

Now, Polarization $P = \varepsilon_0 (\varepsilon_r - 1) E$

and $P = N\alpha_e E.$

Where α_e is electromic polarizability of Helium atom

From above two equation, we can write $N\alpha_e = \varepsilon_0 (\varepsilon_r - 1)$

Or, $\alpha_e = \dfrac{\varepsilon_0 (\varepsilon_r - 1)}{N}$

$= \dfrac{8{\cdot}854 \times 10^{-12}(1{\cdot}0000685 - 1)}{2{\cdot}7 \times 10^{24}}$

Hence $\alpha_e = 2{\cdot}245 \times 10^{-42} Fm^2$

Again $\alpha_e = 4\pi\varepsilon_0 R^3$

Where R is the radius of Helium atom $R = \left(\dfrac{\alpha_e}{4\pi\varepsilon_0}\right)^{1/3} = \left[\dfrac{2{\cdot}245 \times 10^{-42}}{4 \times 3{\cdot}14 \times 8{\cdot}854 \times 10^{-12}}\right]^{1/3}$

R = $\cdot 272 \times 10^{-10}$ meter.

Radius of the Helium atom R = $\cdot 272 \times 10^{-10}$ meter

2. An electric field intensity of strength 10 kV/m is applied across a parallel plate capacitor filled with dietectric constant 2·5 The distance between the plate is 2 mm calculate.

(a) D, (b) P

(c) The surface density of free charge on the plates

(d) The surface density of polarization charge

(e) The potetial difference between the plates

Solution :

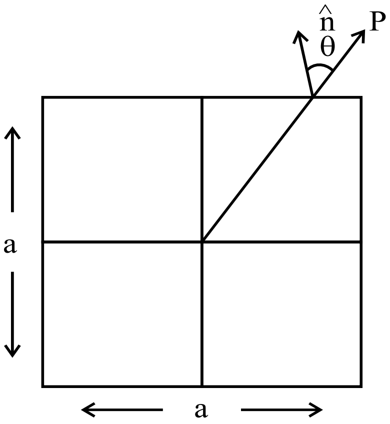
(a) $D = \varepsilon_0 \varepsilon_r E = \frac{10^{-9}}{36\pi} \times 2{\cdot}5 \times 10^4 = 220{\cdot}98\ nc/m^2$

(b) $P = \chi_e \varepsilon_0 E = 1{\cdot}5 \times \frac{10^{-9}}{36\pi} \times 10^4 = 132{\cdot}58\ nc/m^2$

(c) $\rho_s = \vec{D}{\cdot}\hat{i} = D_n = 220{\cdot}98\ nc/m^2$

(d) $\rho_{ps} = \vec{P}{\cdot}\hat{i} = 132{\cdot}58\ nc/m^2$

(e) V = E.d = $10^4 \times (2 \times 10^{-3})$ = 20 V



3. A dielectric cube of side 'a' centred at the origin carries a polarisation charge $P=(\vec{r})=k\vec{r}$, where K is constant. Find all the bound charges and prove that they all add up to zero.

Solution :

The bound volume charge density is equal to $\rho_b = -\left(\vec{\nabla}{\cdot}\overline{P}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k$

Since the bound volume charge density is constant. The total bound volume charge in a cube is equal to the product of the charge density and the volume

$q_{volume} = -\ 3ka^3$

The surface charge density σ_b is equal to,

$\sigma_b = \overline{P}{\cdot}\hat{n} = k\vec{r}{\cdot}\hat{n}$

Now $\vec{r}{\cdot}\hat{n} = r\cos\theta = \frac{1}{2}a$

$\therefore \sigma_b = k\vec{r}\cdot\hat{n} = \frac{1}{2}ka$

The surface charge density is constant across the surface of the cube

$q_{surface} = \frac{1}{2}ka\left(6a^2\right) = 3ka^3$

Thus total bound charge on the cube is equal to

$q_{total} = q_{volume} + q_{surface}$

$= -3ka^3 + 3ka^3$
$= 0$

4. The sphere of radius R carries a polarisation $\vec{P}\left(\vec{r}\right) = k\vec{r}$

where K is constant, and r is the radius vector from the centre.

(a) Calculate bound charges σ_b and ρ_b

(b) Find the field inside and outside the sphere.

Solution : The unit vector $\hat{n}$ on the surface of the sphere is equal to the radial unit vector.

The bound surface charge is equal to $\sigma_b = \vec{P}\cdot\hat{n}\big|_{r=R} = K\vec{r}\cdot\hat{r}\big|_{r=R} = KR$

The bound volume charge density equal to $\rho_b = -\left(\vec{\nabla}\cdot\vec{P}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k$

First consider the region outside the sphere.

The electric field in this region due to the surface charge is equal to

$$\vec{E}_{surface}\left(r\right) = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2\sigma_b}{r^2}\hat{r} = \frac{KR^3}{\varepsilon_0 r^2}\hat{r}$$

The electric field in this region due volume charge is equal to

$$\vec{E}_{volume}\left(r\right) = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi R^3\sigma_b}{r^2}\hat{r} = -\frac{KR^3}{\varepsilon_0 r^2}\hat{r}$$

Hennce the total electric field outside the sphere is equal to zero.

To find the electric field inside the sphere : the electric field due to surface charge is equal to zero. The electric field due to volume charge is equal to

$$\vec{E}_{volume}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4}{3}\pi r^3 \rho_b}{r^2}\hat{r} = -\frac{Kr}{\varepsilon_0}\hat{r}$$

5. Two vast homogenous isotropic dielectrics are in contact in the plane z = 0. For $z \geq 0,\ \varepsilon_{r_1} = 1{\cdot}5$ and for z < 0 $\varepsilon_{r_2} = 1.$

A uniform electric field $\vec{E}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}\, kV/m$ exists for $z \geq 0.$ (a) Find $\vec{E}_2$ for $z \leq 0,$ (b) The angles E_1 and E_2 makes at the interface, (c) The energy densitics inJ/m^3 in both dielectrics.

Solution : The problem is portrayed in Fig. 1... As, $\hat{k}$ is normal to The boundary plane of two dielectrics, normal components are as follows :

$$E_{\text{ln}} = \vec{E}_1 \cdot \hat{n} = \vec{E} \cdot \hat{k} = 4$$

$$\vec{E}_{\text{ln}} = 4\hat{k}$$

$$\vec{E}_{2n} = \left[\vec{E}_2 \cdot \hat{k}\right]\hat{k}$$

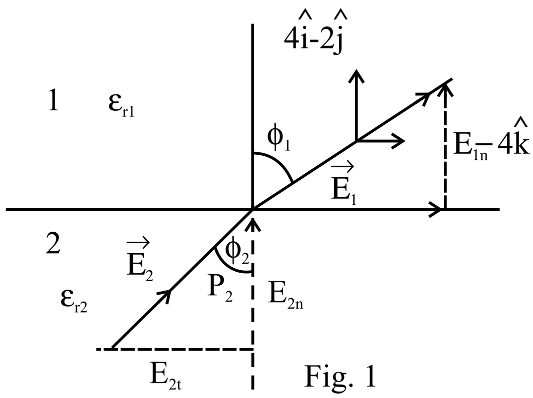


Fig. 1

Also, $\vec{E} = \vec{E}_n + \vec{E}_t$

Hence, tangential component,

$$\vec{E}_{lt} = \vec{E} - \vec{E}_{\text{ln}} = 4\hat{i} - 2\hat{j}$$

Applying boundary condition at the interface we have,

$$\vec{E}_{2t} = \vec{E}_{1t} = 4\hat{i} - 2\hat{j}$$

Similary displacement vector,

$$\vec{D}_{2n} = \vec{D}_{1n} = \varepsilon_{r2}\vec{E}_{2n} = \varepsilon_{r_1}\vec{E}_{1n}$$

$$\vec{E}_{2n} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \vec{E}_{\ln} = \frac{1{\cdot}5}{1}\left(4\hat{k}\right)$$

$$\vec{E}_{2n} = 6\hat{k}$$

Thus electric vector in dielectic $\varepsilon_{r.} = 1$ is

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

$$= \left(4\hat{i} - 2\hat{j} + 6\hat{k}\right) kv/m$$

(b) Let ϕ_1 and ϕ_2 be The angles $\vec{E}_1$ and $\vec{E}_2$ make with the interfacing surface as shown in Fig. 1, while θ_1 and θ_2 are the angles they make to the interface as in figure, we have,

$$\phi_1 = 90^{\circ} - \theta_1$$

$$\phi_2 = 90^{\circ} - \theta_2$$

Since $E_{\ln} = 4$ and $E_{lt} = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

$$\tan\theta_1 = \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2}$$

$$\theta_1 = \tan^{-1}\frac{\sqrt{5}}{2} = 48{\cdot}1888$$

$$\phi_1 = 41{\cdot}8112^{\circ}$$

Similarly $\tan\theta_2 = \dfrac{E_2 t}{E_{2n}} = \dfrac{2\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$

$$\theta_2 = \tan^{-1}\frac{\sqrt{5}}{3} = 36{\cdot}6991^{\circ}$$

$$\phi_2 = 53{\cdot}3009^{\circ}$$

(c) The energy densties are given by

$$W_{E_1} = \frac{1}{2}\varepsilon_1 |E_1|^2 = \frac{1}{2}\times 1{\cdot}5\times\frac{10^{-9}}{36\pi}(16+4+16)\times 10^6$$

$$= 238{\cdot}7\,\mu J/m^3$$

$$W_2 = \frac{1}{2}\times 1\times\frac{10^{-9}}{36\pi}\left(4^2+2^2+6^2\right)\times 10^{-6}$$

$$= 247{\cdot}57\mu\, J/m^3$$

6. An electric field vector $\vec{E} = \left(4\hat{i}+2\hat{j}-4\hat{k}\right)mv/m$ is incident at a particular point on the interface between air and conducting surface. Find $\vec{D}$ and ρ_s at that point.

Solution :

Electric displace ment vector is given by

$$\vec{D} = \frac{10^{-9}}{36\pi}\left[4\hat{i}+2\hat{j}-4k\right]\times 10^{-3}$$

$$= \left[0{\cdot}353\hat{i}+0{\cdot}1768\hat{j}-0{\cdot}353\hat{k}\right]pc/m^2$$

$$\rho_s = |D| = \frac{10^{-9}}{36\pi}\sqrt{\ }\left[4^2+2^2+4^2\right]\times 10^{-3}$$

$$= {\cdot}00593\,pc/m^2$$

7. A sphere of radius R has a dielectric constant ε_r and uniform charge density of ρ_0

(a) Find the potential at the centre of the sphere.

(b) Find the potential at the surface of the sphere.

Solution :

Given $\rho_v = \begin{cases} \rho_O & O < r < R \\ O & r > R \end{cases}$

For r < R, $\varepsilon E_r \left(4\pi r^2\right) = \rho_0 \frac{4\pi r^3}{3}$

Or, $E_r = \frac{\rho_0 r}{3\varepsilon}$

Hence potential, at any point inside the sphere $V = -\int E.dr = -\frac{\rho_0 r^2}{6\varepsilon} + c_1$

For r > R, $\varepsilon_0 E_r \left(4\pi r^2\right) = \rho_0 \frac{4\pi R^3}{3}$

Or, $E_r = \frac{\rho_0 R^3}{3\varepsilon_0 r^2}$

Potential at external point out side the sphere, $V = -\int E.dr = \frac{\rho_0 R^3}{3\varepsilon_0 r} + c_2$

As $r \to \infty$, V = 0 and C_2 = 0

At r = R, $V\left(\vec{r}\right) = V\left(\vec{R}\right)$

$$-\frac{\rho_0 R^2}{6\varepsilon_0 \varepsilon_r} + c_1 = \frac{\rho_0 R^2}{3\varepsilon_0} \Rightarrow c_1 = \frac{\rho_0 R^2}{6\varepsilon_0 \varepsilon_r}\left(2\varepsilon_r + 1\right)$$

(a) So the potential at the centre, V (r = 0)

V (r = 0) $c_1 = \frac{\rho_0 \left(2\varepsilon_0 + 1\right)}{6\varepsilon_0 \varepsilon_r} R^2$

(b) At r = R The surface of the sphere, $V\left(r = R\right) = \frac{\rho_0 R^2}{3\varepsilon_0}$

8. A sphereical shell is filled with dielectric material $\varepsilon = \varepsilon_0 \varepsilon_r$ for a < r < b and ε_0 for 0 < r < a. If a charge q is placed at the centre of the shell, find,

(a) $\vec{P}$ for a < r < b

(b) $\rho_{\rho v}$ for a < r < b

(c) ρ_{ρ} at r = a, and r = b

Answer : Applying Gauss's law,

$$E_r = \begin{cases} \dfrac{Dr}{\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 r^2} b < r < a \end{cases}$$

$$\frac{D_r}{\varepsilon} = \frac{q}{4\pi\varepsilon r^2} \text{ for a < r < b}$$

(1) Now, Polarization is given by,

$$P = \frac{\varepsilon_0 - 1}{\varepsilon_r} D$$

Hence, $P_r = \dfrac{(\varepsilon_r - 1)}{\varepsilon_r 4\pi r^2} q$ for a < r < b

(2) $\rho_{\rho v} = -\vec{\nabla}\cdot\vec{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 P_r\right) = 0$

Surface density of charge, $\rho_{\rho s} = \vec{P} = \left(-\hat{n}_r\right) = +\dfrac{q}{4\pi a^2}\left[\dfrac{\varepsilon_r - 1}{\varepsilon_r}\right] \tan r = a$

$$\rho_{\rho s} = \vec{P}\cdot\left(-\hat{n}_r\right) = -\frac{q}{4\pi b^2}\left[\frac{\varepsilon_r - 1}{\varepsilon_r}\right] for \text{ r} = b$$

9. The electric polarizability of Ar atom is $1{\cdot}7 \times 10^{-40}$ F/m^2. What is the dielectric constant of solid Ar if its density is $1{\cdot}8$ g/cm^3

Solution :

Relative atomc mass of Argan at to atom is 39·95 gm/mole. If N_A is the Avaggadra's member. If N is the no of atom per cc.

$$N = \frac{N_A d}{M_{mol}} = \frac{\left(6{\cdot}02\times10^{23} mol^{-1}\right)\left(1{\cdot}8 cm/cm^{-3}\right)}{\left(39{\cdot}95 gm\ mol^{-1}\right)}$$

$N = 2{\cdot}71 \times 10^{22}\ cm^{-3}$

with, $N = 2{\cdot}71 \times 10^{28}\ m^{-3}$ and $\alpha_e = 1{\cdot}7 \times 10^{-40} Fm^{-1}$

$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_0} = 1{\cdot}55$$

If we use the clausius-Mossdti equation we get $\varepsilon_r = \dfrac{1+\dfrac{N\alpha_e}{3\varepsilon_0}}{1-\dfrac{2N\alpha_e}{3\varepsilon_0}} = 1{\cdot}87$

10. At normal pressure and temperature of 35ºc dielectric constant is k = 1·000516 for Argon atom. Find atomic polarisability and volome.

Solution :

At normal pressure and temperature (0ºc) atomic density is $2{\cdot}687 \times 10^{25}/m^3$

So density at 35ºc is $n = 2{\cdot}687 \times 10^{25} \times \dfrac{273}{273+35}$

$$n = 2.382 \times 10^{25} \Big/ m^3$$

From Clausius Mossolti equation,

$$\alpha = \frac{3\varepsilon_0}{n}\frac{k-1}{k+1} = \frac{3.854\times10^{-12}}{2.382\times10^{25}} \times \frac{.000516}{3.000516}$$

$$= 1.917 \times 10^{-4}\,Fm^2$$

Molecular volume is given by,

$$\frac{4}{3}\pi a^3 = \frac{1}{n}\frac{k-1}{k+2} = \frac{1}{2.382\times10^{25}} \times \frac{.000516}{3.000516}$$
$$= 7.221 \times 10^{-3}\,m^3$$

11. A diaelectric cube of side 'l' centred at the origin carries a frozen-in-polarization $\vec{P} = c\vec{r}$. Find all the bound charges and total charges.

Solution :

Given, $\vec{P} = c\vec{r} = c(x\hat{i} + y\hat{j} + z\hat{k})$

Bound volume charge density

$\rho_b = -\vec{\nabla}.\vec{P} = -c(1+1+1) = -3c$

Total volume charge of the cube $= -3c\ell^3$

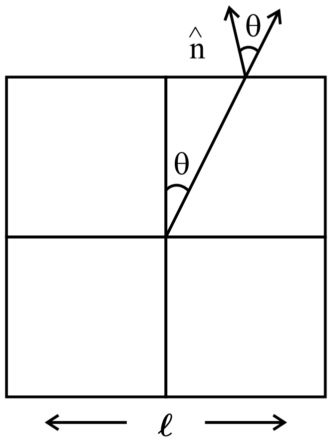


Fig. 2

Bound surface charge density, at the top,

$\sigma_b = \vec{P}.\vec{n} = \frac{c}{2}\ell$ (on all six surfaces)

Hence at the top of the cube total charge C along +ve Z direction,

$= \sigma_b \times \ell^2 = \frac{c}{2}\ell^3$

$\vec{r}\cdot\vec{n} = r\cos\theta = \frac{1}{2}l$

Hence $\sigma_b = c\vec{r}\cdot\hat{n} = \frac{1}{2}cl$

The total surface charges comprising all the six faces of the cube of are l^2 is equal to

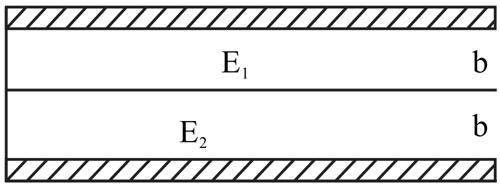
$$q_{surface} = \frac{1}{2}cl \times \left(6l^2\right) = 3cl^3$$

Total bound charges compring volume and surfaces

$q = q_{surface} + q_{volume} =$

$= 3cl^3 - 3cl^3 = 0$

12. The space between two parallel plate copacitor is filled with two slabs of linear dielectric naterial as shown in Fig. 3. Each slab has thickness b, so that the total distance between two plates is 2b. Slab 1 has a dieclectric material of dielectric constant ε_1 and slab 2 has dielectric constant ε_2. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$

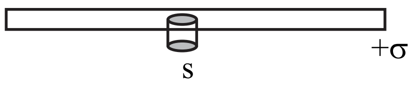


(a) Find the electric displacement Dineach slab.

(b) Find the electric field E in each slab.

(c) Find the polarization in each slab

(d) Find the potential difference between the plate



(e) Find the location and amount of all bound charges

(f) Now knowning all charges recalculate the field in each slab

(a) Applying Guass's law $\int \vec{D} \cdot \vec{ds} = \left(Q_{free}\right)$ enclosed From the Gaussians surface we get.

$DS = \sigma s \Rightarrow D = \sigma$

Note that D = 0 in the metal.

Similarly for the second slab $D = -\sigma$

(b) $\vec{D} = \varepsilon\vec{E} \Rightarrow E = \sigma/\varepsilon_1$ in slab 1, $E = \frac{\sigma}{\varepsilon_2}$ in slab 2.

Again we, know $\varepsilon = \varepsilon_0\varepsilon_r$,

So, $\varepsilon_1 = 2\varepsilon_0$ and $\varepsilon_2 = 1{\cdot}25\varepsilon_0 = \frac{5}{4}\varepsilon_0$

$E_1 = \dfrac{\sigma}{2\varepsilon_0}$ and $E_2 = \dfrac{4\sigma}{5\varepsilon_0}$

(c) $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ so, $\vec{P} = \varepsilon_0 \chi_e \dfrac{\sigma}{\varepsilon_0 \varepsilon_r} = \left(\dfrac{\chi_e}{\varepsilon_r}\right)\sigma$

Now, $\chi_e = \varepsilon_r - 1$,

$\Rightarrow P = (1 - 1/\varepsilon_r)\sigma = \left(1 - \varepsilon_r^{-1}\right)\sigma$

$P_1 = \sigma/2$ and $P_2 = \dfrac{\sigma}{5}$

(d) Now potential $V = E_1 b + E_2 b = \dfrac{b\sigma}{\varepsilon_0}\left[\dfrac{1}{2} + 4/5\right]$

$= \dfrac{b\sigma}{\varepsilon_0}\dfrac{13}{10}$

(e) Volume charge density $\rho_b = 0$

Now bound charges is slabs :

$\sigma_b = +P_1$ at the bottom of slab $1 = \sigma/2$

$\sigma_b = -P_1$ at the bottom of slab $1 = -\sigma/2$

$\sigma_b + P_2$ at the bottom of slab $2 = \dfrac{1\sigma}{5}$

$\sigma_b = -P_2$ at the bottom of slab $2 = -\dfrac{\sigma}{5}$

(f) In slab 1 total surface charge above $\sigma - \sigma/2 = \sigma/2$

total surface charge below $= \sigma/2 - \sigma/5 + \dfrac{\sigma}{5} - \sigma = -\sigma/2$

which implies $E_1 = \dfrac{\sigma}{2\varepsilon_0}$

Inslab 2 total surface charge above $0 - \dfrac{\sigma}{2} + \dfrac{\sigma}{2} - \dfrac{\sigma}{5} = \dfrac{4}{5}\sigma$

total surface charge below $\dfrac{\sigma}{5} - \sigma = -\dfrac{4\sigma}{5}$

which implies $E_2 = \dfrac{4\sigma}{5\varepsilon_0}$

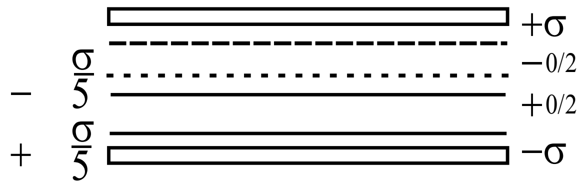


Fig. 4

UNIT 3 : Magnetic Field

3.1 Objective

After completing this unit you will be able to understand–

1. The force due to magnetic field over a moving charge and trajectory of charge in a magnetic field.

2. The origin of magnetic field due to flow of charge through two laws, Biot-Savart's law and Ampere's law.

3. Application of Biot-Savart's law to find magnetic induction for

a) straight current carrying finite and infinite one dimensional conductor.

b) circular loop,

c) solenoid.

4. Application of Ampere's law to find the magnetic field in some symmetric cases of current distribution.

5. Vector magnetic potential.

3.2 Introduction

The history of magnetic effect has been known from the ancient time; from the discovery of loadstone. The relation between electricity and magnetism was first established experimentally by Oersted in 1819, though such connection had been hinted in a book by Gilbert in 1600. The quantitative relation between current and magnetic field was established by Biot-Savart and Ampere during the period 1820-1825.

3.3 A brief recapitulation on Magnetism

We have already come across the existence of magnetic field which is prouduced by permanent magnet or by moving electric charge and so they also under go magnetic interaction when placed in a magnetic field according to Newtonian law of action and reaction. The magnetic field is described by magnetic field lines which provides its direction (along the tangent to the field line at the point concerned) and the magnetic field induction $\vec{B}$ at a point is the number of field lines corssing per unit area through the point, when the area is held perpendicular the field lines at the point concerned. Unlike the electric field lines the magnetic field lines are closed which leads to the conclusion of nonexistence of free magnetic poles in nature.

The force on a moving charge q in electric field $\vec{E}$ and magnetic induction $\vec{B}$ is given Lorentz force, $d\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

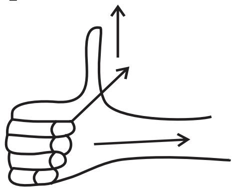
Thus the force on a moving charge due to static magnetic field

$$d\vec{F}_m = q\vec{v} \times \vec{B} \ldots\ldots(3.1.1)$$

So both static and moving charge experiene electric interaction but only a moving charge may undergo magnetic interaction. The direction of force is perpendicular to both $\vec{v}$ and $\vec{B}.$ The magnetic force is a no work force as

$$dw = \vec{F}.d\vec{r} = \vec{F}.\frac{d\vec{r}}{dt}dt = \vec{F}.\vec{v}\,dt = 0.$$

Fig. 3.1



The direction of the force can be obtained easily by right hand thumb rule shown in fig (3.1). The thumb finger gives the direction of force when the other fingers specify the direction of rotation from $\vec{v}$ to $\vec{B}.$

3.4 The Unit of $\vec{B}.$

The magnetic field induction at a point is said to be unity if 1C of charge moving at 1m/s, perpendicular to the field, experiences a force of 1 Newton. This unit is called Tesla.

$$1N = 1C1ms^{-1}B\sin(\pi/2)1T$$

$$T = \frac{N}{Cms^{-1}} = \frac{N}{Am}$$

3.5 Track of charged particle in uniform magnetic field

1. When $\vec{v}$ *is perpendicular to* $\vec{B}$

Here F = qvB sin 90º = qvB and is directed along $\vec{v}\times\vec{B}.$ So the motion is confined in a plane with radius of rotation r, such that centripetal force qvB = mv^2/r, r r = mv/qB. If ω is the is the angular frequency of rotation, then v = ωr, or r = m ωr/qB or ω = qB/m, so T = 2πm/qB. Thus we see that the time period of rotation is independent of velocity of particle and depends on the q/m of the particle. Motion is shown in fig. (3.2)

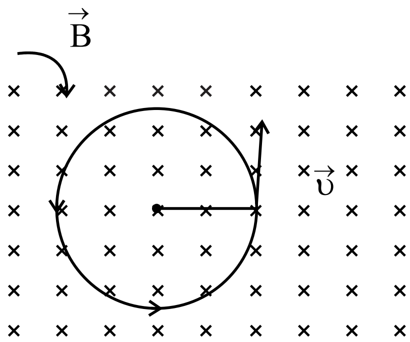


Fig. 3.2

2. When $\vec{v}$ makes an angle θ with the direction of $\vec{B}.$

In this case the term v cosθ remains unaffected by magnetic field, since this component is along the line of $\vec{B}.$ For the component v sin θ the magnetic force F_m = qvB sinθ =

$mv^2 \sin^2 \theta/r$, or $r = mv\sin\theta/qB$. So the time period of rotation $T = 2\pi\, m/qB$. The motion is as shown in fig (3.3)

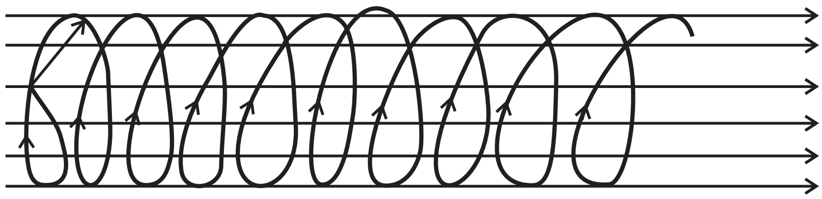


Fig. 3.3

So the path is helical with pitch (= the distance moved in the net direction of motion in each rotation = $v\cos\theta\,(T) = 2\pi mv\cos\theta/qB$)

The force on a current carrying conductor in a uniform magnetic field.

we consider a segment *dl* of current carrying conductor carrying current *i* as shown in fig. (3.4)

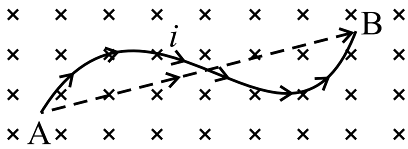


Fig. 3.4

dq be the charge flowing through the element *dl* in time *dt.* Then the force on the element

$$d\vec{F}_m = dq\vec{v}\times\vec{B} = dq\frac{d\vec{l}}{dt}\times\vec{B} = \frac{dq}{dt}d\vec{l}\times\vec{B} = id\vec{l}\times\vec{B}$$

$$d\vec{F}_m = id\vec{l}\times\vec{B} \;\ldots\ldots(3.5.1)$$

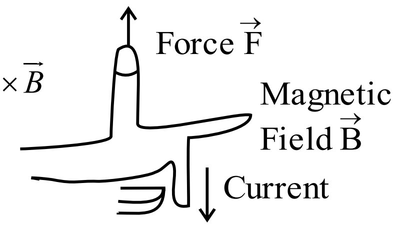


Fig. 3.5

So the force is perpendicular to both $d\vec{l}$ and $\vec{B}$, the direction of force can easily be obtained from Fleming's left hand rule as in fig. (3.5).

The total force on the conductor

$$\vec{F}_m = \int id\vec{l}\times\vec{B} = i\int d\vec{l}\times\vec{B} = i\vec{L}\times\vec{B} = i\vec{AC}\times\vec{B} \;\ldots\ldots(3.5.2)$$

3.6 Biot-Savart Law

So far we have discussed the force on charge or current element in magnetic field. Now we shall discuss how the current produce the magnetic field. This is given by Biot-Savart law.

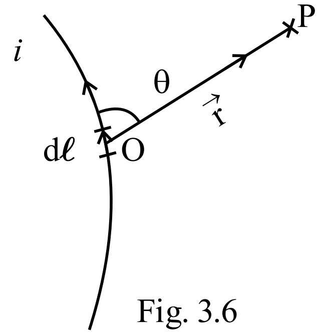


Fig. 3.6

The law states that if i be the current through an element length $\vec{dl}$ then the magnetic field produced by the current element $i\vec{dl}$ at position vector $\vec{r}$ with current element as origin in SI unit is given by (please refer the fig. (3.6)

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{\vec{dl} \times \vec{r}}{r^3} \quad \ldots\ldots(3.6.1)$$

The direction of magnetic field at a point can be obtained from $\vec{dl} \times \vec{r}$ or by right hand thumb rule. It states that if we stretch our right thumb in the direction of current element and curl our other fingers through p specify the direction of magnetic field.

Application of Biot-Savart Law.

1) *Magnetic field intensity due to a straight current carrying conductor.*

The fig. (3.7) shows a current carrying conductor with current *i.* To calculate magnetic field intensity at the point p at a distance a from the conductor we take an current element $i d\vec{y}$ at point y as in fig. (3.7). The magentic field due to this elemental current at P, using

Biot-Savart law $d\vec{B} = \frac{\mu_0}{4\pi} i \frac{\vec{dy} \times \vec{r}}{r^3}$

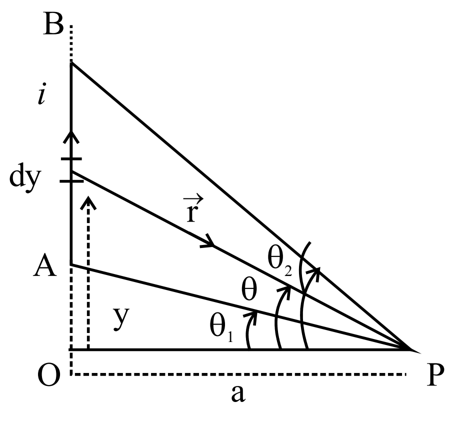


Fig. 3.7

$dB = \frac{\mu_0}{4\pi} i \frac{dy \cos\theta}{r^3}$ directed vertically inside the plane of paper.

So the total magnetic field intensity at P

$$B = \frac{\mu_0}{4\pi} i \int_{\theta_1}^{\theta_2} \frac{dy \cos\theta}{r^2}$$

We put $y = a\tan\theta$, Then $dy = a\sec^2\theta d\theta$. Again $a/r = \cos\theta$

Thus $B = \frac{\mu_0 i}{4\pi}\int_{\theta_1}^{\theta_2}\frac{a\sec^2\theta d\theta\cos\theta\cos^2\theta}{a^2} = \frac{\mu_0 i}{4\pi a}\int_{\theta_1}^{\theta_2}\cos\theta d\theta$

Or, $B = \frac{\mu_0 i}{4\pi a}(\sin\theta_2 - \sin\theta_1)$, ... (3.6.2) acting down The plane of paper.

Thus we see that at a distance *a* from the wire the field is constant in magnitude and is tangent to the circle of radius *a* following right hand cork-screw rule and the field lines are closed.

For a straight infinitely long conductor $\theta_2 \to \pi/2$ *and* $\theta_1 \to -\pi/2$.

So from eqn. (3.6.2) we have

$\vec{B} = \frac{\mu_0}{4\pi a}2i$ ... ...(3.6.3) Vertically inside the plane of paper.

2. *Field due to a current carrying circular loop.*

The fig. (3.8) shows a circular loop of radius *a* carrying current *i* in anticlockwise direction when viewed from the right side of the fig. (3.8). We have to find out the magnetic field intensity B at a point P on the axis of the coil at a distance x from the centre O of the ring.

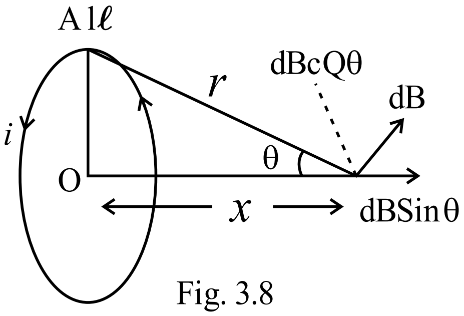


Fig. 3.8

We consider an elemental length *dl* at A. Then magnetic field at p due to the current element $i\vec{dl}$ at A is $d\vec{B} = \frac{\mu_0}{4\pi}i\frac{dl}{r^2}$ along *PC*

(Since $d\vec{B}$ is perpendicular to both $\vec{r}$ and the ring wire at A)

Resolving $d\vec{B}$ along op and perpendicular to op as $dB\sin\theta$ and $dB\cos\theta$, we see that, the component $dB\cos\theta$ vanishes on summation over the entire wire by symmetry So the neat field is along OP and is,

$B_H\int\frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{i2\pi a}{r^2}\frac{a}{r}$

$$= \frac{\mu_0}{4\pi} \frac{2\pi a i}{\left(a^2 + x^2\right)^{3/2}} = \frac{\mu_0}{4\pi} \frac{2\pi a^2 i}{\left(a^2 + x^2\right)^{3/2}}$$

So, $\vec{B}_H = \frac{\mu_0}{4\pi} \frac{2\pi a^2 i}{\left(a^2 + x^2\right)^{3/2}}$ along OP.

For a coil with Nturens

$$\vec{B}_H = \frac{\mu_0}{4\pi} \frac{2\pi a^2 N i}{\left(a^2 + x^2\right)^{3/2}} \quad \text{along OP ... ... (3.6.4)}$$

3.7 Torques and Forces in Magnetic Dipole

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole in an electric field. Let us calculate The torque on a rectangular current loop in a uniform magnetic field. Loop is placed at the origin and at inclination of θ from The z-axis towards the y axis. Let $\vec{B}$ point in the z-direction. The forces on the two sloping sides cancel. The forces on the horizantal sides likewise equal and opposite, so the net force is zero, but they form a torque $\vec{N} = aF \sin\theta \hat{i}$

The magnitude of the force on each of these segment is F = IbB

and therefore $\vec{N} = Iab \sin\theta \hat{i}$

Or, $\vec{N} = \vec{m} \times \vec{B}$ ... ...(3.7.1)

Where m = Iab is the dipole moment of the loop.

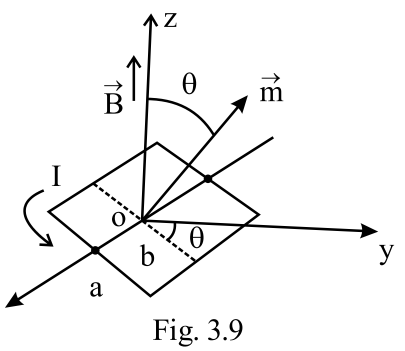


Fig. 3.9

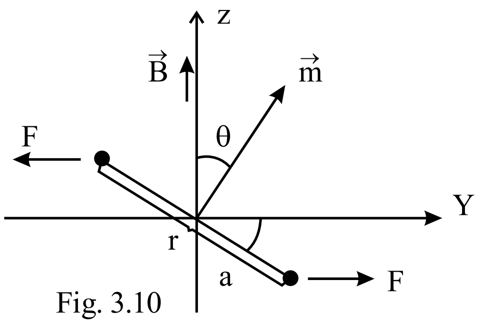


Fig. 3.10

The torque $\vec{N}$ tends to dearease the angle θ, If U is the potential energy of the loop.

Then, $\frac{\partial U}{\partial \theta} = N = mB \sin\theta$

so that $U = -mB\cos\theta + \text{constant}$. Taking $U = 0\ at\,\theta = \pi/2$. we get

$U = -mB\cos\theta = -\vec{m}\cdot\vec{B}$ ... ...(3.7.2)

The equation (3.7.2) is similar to the expression for the potential energy of an electric dipole of moment $\vec{P}$ placed in an electric field. $\vec{E}$ is $U = -\vec{P}\cdot\vec{E}$

3.8 Ampere's Law

The Ampere's law states that the line integral of magnetic field vector about a closed path is equal to μ times the current through the surface enclosed by the closed path, mathematically

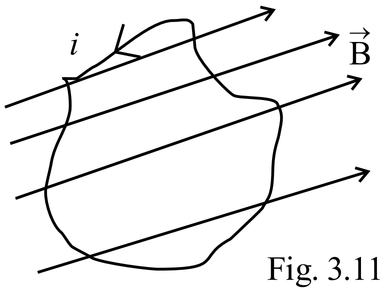


Fig. 3.11

$\oint \vec{B}.d\vec{l} = \mu_0 I$ ... ...(3.8.1)

were μ_0 stands for the permeability of of free space, in case of medium it should be replaced by permeability of the medium. If $\vec{J}$ is the current density then, $I = \int_S \vec{J}.d\vec{s}$,

(where $d\vec{s}$ stands for an elemental surface area)

So the Amere's law takes the form,

$$\oint \vec{B}.d\vec{l} = \mu_0 I = \mu_0 \int_S \vec{J}.d\vec{s}$$

Using Stoke's theorem $\oint \nabla\times\vec{B}.d\vec{s} = \mu_0 I = \mu_0 \int_S \vec{J}.d\vec{s}$

Since this equation is true for all value of $\vec{s}$ so,

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \ldots\ldots(3.8.2)$$

Application of Ampere's law

1) *Field at a point due to a long current carrying conductor*

The fig. (3.12) shows long current carrying conductor carrying current i. We have to find out the expression of magnetic field intensity B at point P at a distance a from the conductor as in fig. (3.12). In the mode of calculation we draw a circle of radius a around the conductor, then by symmerty field on it wil be of sme magnitude and acting tangent to the circle. (Direction of field is given by Biot-Savart law)

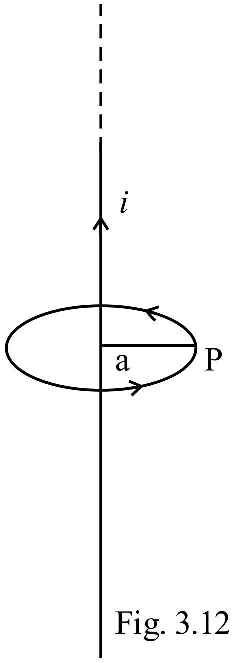


Fig. 3.12

Now by Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i$$

Or, $B 2\pi a = \mu_0 i$

Or, $B = \dfrac{\mu_0}{2\pi a} \quad \ldots\ldots(3.8.3)$

2) *Field due to a long solenoid*

We consider a solenoid of n-turns per unit length. The length of the solenoid be very very long compared to its radius. i be the current flowing through it in a anticlockwise direction when viewed from the right. The fig. (3.13) shows the solenoid and fig. (3.14) shows a vertical section of The solenoid.

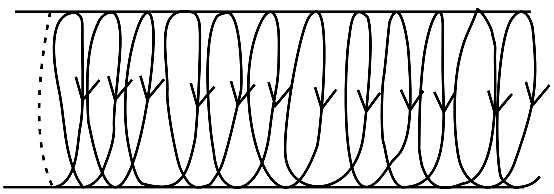


Fig. 3.13

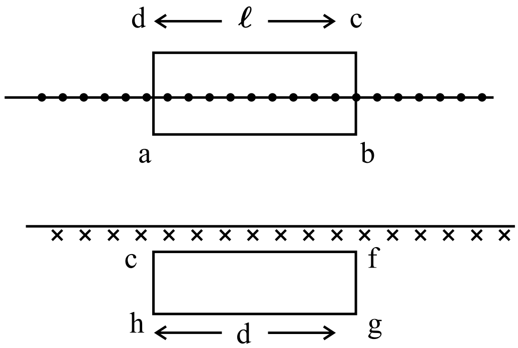


Fig. 3.14

We draw two loops *abcd* and *efghe* each of length *l* as shown in fig. (3.14) We consider the loop *efghe*. Here

$$\oint_{efghe} \vec{B}.d\vec{l} = 0 \text{ as no current is entrapped in the loop.}$$

As this valid for any such loop so B = 0.

Thus the magnetic field intensity outside the loop is zero.

Now we take the loop *abcda*

$$\oint_{abcda} \vec{B}.d\vec{l} = \mu_0 inl$$

$$\text{Or, } \int_a^b \vec{B}.d\vec{l} + \int_b^c \vec{B}.d\vec{l} + \int_c^d \vec{B}.d\vec{l} + \int_b^a \vec{B}.d\vec{l} = \mu_0 nil$$

$$\text{Or, } Bl + O + O + O = \mu_0 nil$$

$$\therefore B = \mu_0 ni \; \ldots\ldots (3.8.4)$$

Here we have cosidered the field inside the solenoid is axial and uniform.

3) *Field due to a toroid*

A toroid is a device consisting of a ring (a torus) wrapped with insulated conducting wire.

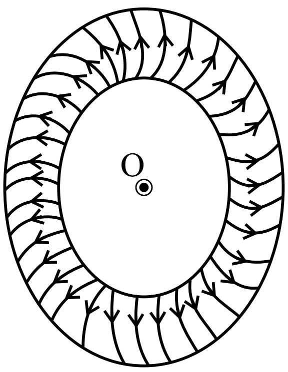


Fig. 3.15

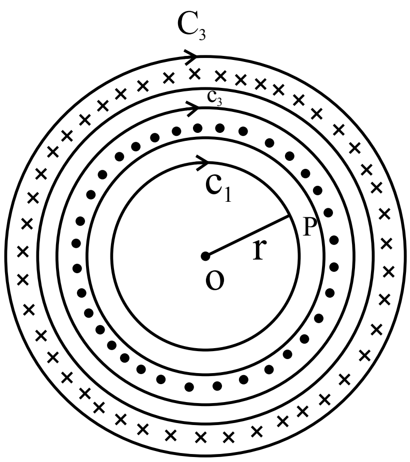


Fig. 3.16

The fig. (3.15) shows a toroid of inner radius *a* and outer radius *b* with n turns per unit length, carrying current *i*. The fig. (3.16) shows a section of the toroid in a plane through the plane of the ring passing through its Centre O of the ring.

To find the magentic field intensity at a point P at a distance *r* from centre please refer fig. (3.16).

i) When $r > a$, $\oint \vec{B}.d\vec{l} = 0$ *as no current is entrapped in the loop.* So B = 0.

ii) For $r > b$, $\oint \vec{B}.d\vec{l} = 0$ as current enclosed in the loop is again zero, since each turn passes twice throgh it carrying equal but opposite current.

iii) For $a < r > b$, within the core of the toroid

$$\oint \vec{B}.d\vec{l} = B \times 2\pi r = \mu_0 Ni \text{ according to ... Amper's law.}$$

So, $B = \mu_0 \, Ni/2\pi r = \mu_0 ni.$ ... ... (3.8.5)

Thus we see the field inside the core of a toroid is not constant. However the field reamains fairly constant when the inner radius and outer radius are close to each other.

4. ***Magnetic field due to a current carrying cylinder***

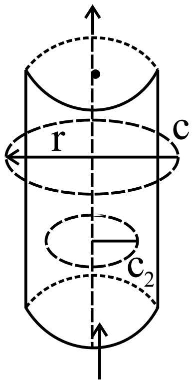


Fig. 3.17

The fig. (3.18) shows a long cylinder carrying current *i*. J be the current density which is taken to be constant. From symmetry, field lines are closed circles, co-centric with the axis of wire. We are calculating magnetic inducution at point P at a distance *r* from Tha axis.

(a) When $r > a$, P is outside the wire. We consider the dotted loop C_1. $\vec{B}_0$ be the magnetic inducution on C_1. Then $\oint_{C_2} \vec{B}.\vec{dl} = B_0 2\pi r = \mu_0 i = \mu_0 i \quad B_0 = \mu_0 i/2\pi r$ ... ...(3.13)

(b) When P is inside The cylinder $r < a$. Refer to loop C_2. $\oint_{C_2} \vec{B}_2 \cdot d\vec{l} = \mu_0 J \pi r^2$

$$\Rightarrow B_i 2\pi r = \mu_0 J \pi r^2 \frac{a^2}{a^2} = \mu_0 i r^2 / a^2 \quad \therefore B_i = \frac{\mu_0 i r}{2\pi a^2} \text{ ... ...(3.8.6)}$$

5. ***Force between two parallel current carrying wires.***

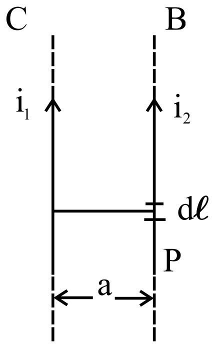


Fig. 3.19

The fig. (3.19) shows two long Parallel current carrying conductors A and B, separated by a distance a, carrying current i_1 and i_2 respectively. Then using Ampere's law the magnetic induction $\vec{B}$ at point P due to the conductor A is,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2i_1}{a}, \text{ directed normally into the plane of paper.}$$

The force on a current element $i_2 d\vec{l}$ at P due to this $\vec{B}$ is

$$d\vec{F} = i_2 d\vec{l} \times \vec{B}$$

$d\vec{F} = i_2 dlB$ acting along PQ

$= \frac{\mu_0}{4\pi}\frac{2i_i i_2}{a} B$ dl acting along PQ

So the force per unit length on wire B is $\frac{\mu_0}{4\pi}\frac{2i_1 i_2}{a} B$ along PQ ... ... (3.8.7)

Now using Netwon's third law the force per unit length on A due to current in B will be same but will act along B. So the wires will attract each other. If the currents are oppositely directed the wires will repel each other.

3.9 Properties of $\vec{B}$

To know a vector completely, we must explore its curl and divergence (Helmholtz criterion).

We have already explored the curl of $\vec{B}$ through Ampere's law as $\nabla \times \vec{B} = \mu_0 J$

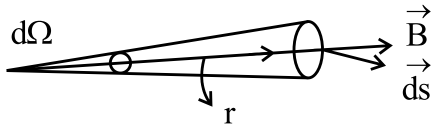


Fig. 3.20

Now to explore $\nabla.\vec{B}$, we define magnetic flux through an elemental surface ds, as the number of field lines passing through the surface perpendicular to the surface. The magnetic field intensity B at a point is defined as the number of field lines passing through the point per unit area, when the area is placed perpendicular to the direction of the field. So magnetic flux through an elemental area *ds,* at a point where the magnetic field intensity vector is $\vec{B}$ as shown in the fig. (3.20)

$$d\phi = \vec{B}.d\vec{s}$$

So the total magnetic flux through a surface S, $\phi = \oint_S \vec{B}.d\vec{s}.$ Now for a closed surface $\phi = \oint_S \vec{B}.d\vec{s} = 0,$ since the magnetic field lines are closed, as isolated pole does not exist in nature.

Now using Gauss's divergence theorem we can write.

$\phi = \oint_S \vec{B}.d\vec{s} = \oint_V \nabla.\vec{B}.dv = 0$ as this is applicate for any volume.

So $\nabla.\vec{B} = 0.$ ... ...(3.9.1)

As the divergence of curl of a vector is always zero, so we can write

$\vec{B} = \nabla \times \vec{A}$ ... ...(3.9.2)

Where $\vec{A}$ is called megnetic vector potential.

It is to be mentioned that that vector potential $\vec{A}$ is not uniquely defined through the equation (3.9.2) as it $\vec{B}$ remains unchanged with addition of a function whose curl is zero.

(1) *Vector magnetic potential for a current loop*

The fig (3.21) shows a current loop carrying current *i.* Then for a current element $id\vec{l}$, the magnetic field at a point P of position vector $\vec{r}$ with current element at origin is given by Biot-Savart law,

$$d\vec{B} = \frac{\mu_0}{4\pi} id\vec{l} \times \frac{\vec{r}}{r^3}$$

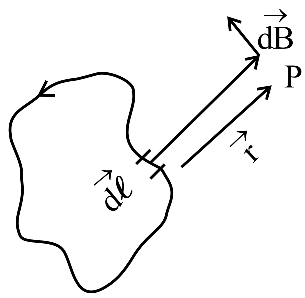


Fig. 3.21

Now $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r} = -\frac{1}{r^3}\vec{r}$

So we can write $d\vec{B} = -\frac{\mu_0}{4\pi}\left(i d\vec{l}\right) \times \nabla\left(\frac{1}{r}\right)$

$$= \frac{\mu_0}{4\pi}\nabla\left(\frac{1}{r}\right) \times \left(i d\vec{l}\right) \quad \ldots\ldots(3.9.3)$$

Now $\nabla \times \left(\phi\vec{A}\right) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A}$, so using this identity in the above equation we have

$$d\vec{B} = \frac{\mu_0}{4\pi}\left[\nabla \times \left(\frac{i d\vec{l}}{r}\right) - \frac{1}{r}\nabla \times \left(i d\vec{l}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\nabla \times \left(\frac{i d\vec{l}}{r}\right), \text{ since } \nabla \times i d\vec{l} = 0$$

$$\therefore\ \vec{B} = \frac{\mu_0}{4\pi}\oint \nabla \times \left(\frac{i d\vec{l}}{r}\right) = \nabla \times \oint \frac{\mu_0}{4\pi}\frac{i d\vec{l}}{r}.$$

Compering this equation with $\vec{B} = \nabla \times \vec{A}$,

$$\therefore \vec{A} = \oint \frac{\mu_0}{4\pi}\frac{i d\vec{l}}{r} \quad \ldots\ldots(3.9.4)$$

Again $\vec{A} = \oint \frac{\mu_0}{4\pi}\frac{\left(\vec{J}\cdot d\vec{s}\right)d\vec{l}}{r} = \int_V \frac{\mu_0}{4\pi}\frac{\vec{J}dv}{r} \quad \ldots\ldots(3.9.5)$

As $\vec{A}$ can't define $\vec{B}$ uniquely, $\vec{A}$ can be considered as a mathematical interstep for computer of $\vec{B}$.

2) Multipole Expansion of the vector potential

In order to find the appromimate value of vector potential due to a localized current distribution, method of multipole expansion of potential can offer appromimate value at large distance from the source, which can be expressed in powers of $1/r$, $1/r^2$. Higher order terms with negligible non-zero value in the series is the one important aspect of this method, ensuring appronimately fair value of potential. We get the expansion as follow from the figure–

$$\frac{1}{r''}=\frac{1}{\sqrt{\left[r^2+r'^{12}-2rr'\cos\theta\right]}}$$

$$=\frac{1}{r}\sum_{n=0}\left(\frac{r'}{r}\right)^n P_n C\cos\theta \text{ ... ...}(3.9.6)$$

Where θ is the angle between $\vec{r}$ and $\vec{r}'$ Accordingly vector potential of a current loop can be written as $\vec{A}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\oint\frac{1}{r''}dl'=\frac{\mu_0 I}{4\pi r^{n+1}}\oint(r')^n P_n(\cos\theta)dl' \text{ ... ...}(3.9.7)$

Or, $\vec{A}\left(\vec{r}\right)=\mathrm{I}\frac{\mu_0}{4\pi}\left[\frac{1}{r}\oint dr'+\frac{1}{r^2}\oint r'\cos\theta\, dl'+\frac{1}{r^3}\oint(r')^2\left(\frac{3}{2}\cos^2\theta-1/2\right)dl' \text{... ...}(3.9.8)\right.$

Now the magnetic monopole term is always zero, for the integral is just the total vector displacement around a closed loop :

$$\oint dl'=0 \text{ ... ...}(3.9.10)$$

Dipole term plays important role as monopole term is zero,

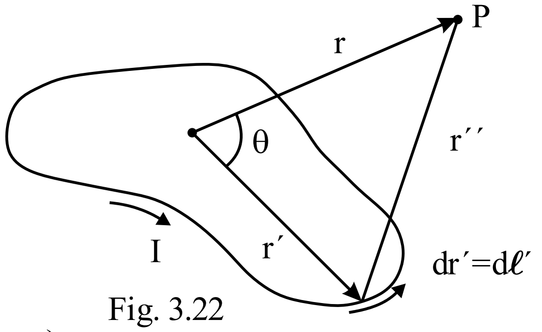


Fig. 3.22

$$\vec{A}_{dipole}\left(\vec{r}\right)=\frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta\, d\vec{l}'=\frac{\mu_0 I}{4\pi r^2}\oint\left(\hat{r}.\vec{r}'\right)d\vec{l}'. \text{ ... ...}(3.9.10)$$

This integral can be written elegant way if we use the following relation–

$$\oint\left(\hat{r}.\vec{r}'\right)d\vec{l}'=-\hat{r}\times\int ds' \text{ ... ...}(3.9.11)$$

Then $\vec{A}_{dip}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \text{ ... ...}(3.9.12)$

where $\vec{m}$ is the magnetic dipole moment $\vec{m}=I\int d\vec{s}=I\vec{s} \text{ ... ...}(3.9.13)$

where $\vec{s}$ is the vector area of the loop. If the loop is flat, $\vec{s}$ is the ordinary area enclosed, with direction followed by the usual right hand rule. In reality, the dipole potential is suitable appronimation whenever the distance r greatly exceeds the size of the loop.

The magentic field of a perfect dipole is easiest way to calculate if we allign $\vec{m}$ at the origin and in the z-direction.

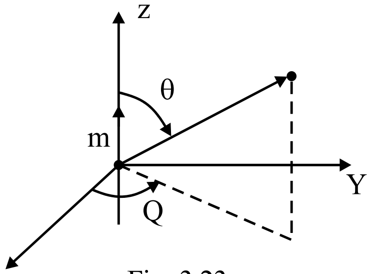


Fig. 3.23

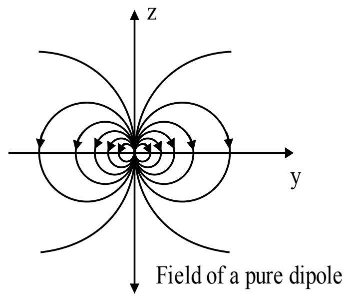


Fig. 3.24

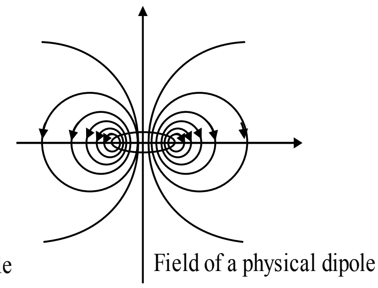


Fig. 3.25

$$\vec{A}_{dip}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}\ \ldots\ \ldots\ (3.9.14)$$

so, $\vec{B}_{dip}\left(\vec{r}\right)=\vec{\nabla}\times\vec{A}=\dfrac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)$

Astonishingly, this is identical in structure to the field of a electric dipole

If we write $\vec{m}=\left(\vec{m}.\hat{r}\right)\hat{r}+\left(\vec{m}\cdot\hat{\theta}\right)\hat{\theta}$

$=m\cos\theta\hat{r}-m\sin\theta\hat{\theta}$

Then $3\left(\vec{m}.\hat{r}\right)\hat{r}-\vec{m}=3m\cos\theta\hat{r}-m\cos\theta\hat{r}+m\sin\theta\hat{\theta}$

$=2m\cos\theta\hat{r}+m\sin\theta\hat{\theta}$

So can write $\vec{B}_{dip}\left(\vec{r}\right)as$

$$\vec{B}_{dip}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\vec{m}.\hat{r}\right)\hat{r}-\vec{m}\right]\ \ldots\ \ldots(3.9.15)$$

Also magnetic dipole can be written in the following way–

we know $\vec{m} = Is = \frac{1}{2}\oint \vec{r} \times \mathrm{I}\, d\ell'$

$$\vec{m} = \frac{1}{2}\int \left(\vec{r} \times \vec{J}\right) d\tau \; \ldots \ldots (3.9.16)$$

3.10 Summary

After studying the unit we should understand following.

1. The magnetic force on a moving charge $\vec{F} = q\vec{v} \times \vec{B}.$ It is a no work force.

2. The trajectory of charge in magnetic field is circular for $\vec{v} \perp \vec{B}$ and helical for other angles of projection.

3. Biot-Savart's law $dB = \frac{\mu_0}{4\pi} i \frac{dl \cos\theta}{r^2}$ and its application.

4. Amperes law $\oint \vec{B}.d\vec{l} = \mu_0 I$ and its application.

5. Study of nature of $\vec{B}.$ Introduction of vector potential $\vec{A}\ as\, \vec{B} = \vec{\nabla} \times \vec{A}$

6. Magnetic dipole $\vec{m} = \frac{1}{2}\int \left(\vec{r} \times \vec{J}\right) d\tau$

7. Forque on a current loop $\vec{N} = \vec{m} \times \vec{B}$

8. Vector potential of magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$

3.11 Review Questions and Answer

1. What represents The line integral of magnetic vector potetial $\vec{A}$ about the boundary of surface in a magnetic field?

2. If the flux density at a point in space is $\vec{B} = x\hat{i} - 2ay\hat{j} + 4\hat{k}\left(a = consant\right),\ \hat{i}, \hat{j}\ and$

k are unit vectors along x, y and z directions. The find the value of *a.*

3. Show that magnetic force on a charge is a no work force.

4. Is megnetic field is a conservative field?

5. A beam of charge undergoes deflection in a space. Can you identify which field electric or magnetic field is present in the field.

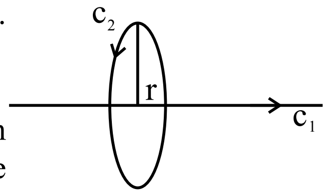


Fig. 3.26

6. A current is sent through a hanging coiled spring. What change do you expect when the current is suitehed off.

7. Two long parallel conducting wires carrying current i_1 & i_2 are kept separated parallel to each other at a distance d. Will the force between the wires increase if the diameter of one wire is donbled.

8. A straight wire carrying current i_1 is placed along the centre of loop carrying current i_2 as shrown in figure. Is there any force of interaction between the coil and straight wire. (Neglect gravitational interaction).

9. Starting from the expression of magnetic vector potential $\vec{A} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}}{R}$ obtain. The expression for magnetic induction $\vec{B}$. Also show that $\vec{\nabla}\cdot\vec{B} = 0$.

10. Find the force between two ideal magnetic dipoles of moments $\vec{m_1}$ and m_2 separated by a distance r. Assume that m_1 and m_2 point in the direction of the vector joining them. (Ch–13)

11. Find the vector potential of inside and out side a sole noid with n turns per unit length givn current I and dradius R

1. The line integral of $\vec{A}$ over a boundary of surface S is given by $\oint \vec{A}.d\vec{l}$.

Now applying stokes law $\oint \vec{A}.d\vec{l} = \int_S \left(\vec{\nabla}\times\vec{A}\right).d\vec{S}$

Now $\vec{B} = \nabla\times\vec{A}$ by definition. $\oint \vec{A}.d\vec{l} = \int \vec{B}.d\vec{S}$ = flux through the surface S.

2. $\vec{B}$ to be a magnetic flux density $\nabla.\vec{B} = 0$.

$$\nabla.\vec{B} = \frac{\partial x}{\partial x} - \frac{\partial(2ay)}{\partial y} + \frac{\partial 4}{\partial z} = 0$$

$1 - 2a = 0$, Or, $a = 1/2$

3. The magnetic force on charge is given by

$\vec{F} = q\vec{v} \times \vec{B}$, now work done for a displacement $d\vec{r}$

$$dw = \vec{F}.d\vec{r} = q\left(\vec{v} \times \vec{B}\right).d\vec{r} = q\left(\vec{v} \times \vec{B}\right)\cdot\frac{d\vec{r}}{dt}dt = q\left(\vec{v} \times \vec{B}\right).\vec{v}dt = 0,$$

as $\left(\vec{v} \times \vec{B}\right)$ is perpendicular to $\vec{v}$. Thus magnetic force is a no work force.

4. No. Please see differential form of Ampere's law.

5. Consult text.

6. When the current flows each spiral attracts the neighbours turn and the coil turns become closer. When current is switched of the distance between the turns increases.

7. No. The force depends on current and mean distance of separation.

8. No. The magnetic field produced by each of them is along the direction of other, So $= \vec{F} = id\vec{l} \times \vec{B} = 0.$

9. Current density is a function of source co-ordinate, while here all the differential operatars act m field co-ordinaltes.

Assuming source co-ordinate as (x', y', z') and field co-ordinates as (x, y, z), so $R = \vec{r} - \vec{r}'$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left[\frac{\mu_0 I}{4\pi}\right]\frac{dl}{R}\left[\because\ Id\vec{l} = \vec{J}(r)d\tau\right.$$

$$= \vec{\nabla} \times \frac{\mu_0}{4\pi}\int\frac{\vec{J}d\tau}{R}\ \ldots\ldots(1)$$

Now, $\vec{\nabla} \times \left(\frac{\vec{J}}{R}\right) = \vec{\nabla}(1/R) \times \vec{J}\left(\vec{r}'\right) + \frac{1}{R}\vec{\nabla} \times \vec{J}\left(\vec{r}'\right)$

Now, $\vec{\nabla}\times\vec{J}(r')=0$ as $\vec{\nabla}$ is an operator of $f(r)=f(x,y,z)$

$$\therefore \vec{\nabla}\times\frac{\vec{J}}{R}=\frac{\vec{J}\times\vec{R}}{R^3} \quad \ldots\ldots(2)$$

From (1) and (2)

$$\vec{B}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}\times\vec{R}}{R^3}d\tau$$

Now $\vec{\nabla}.\vec{B}=\frac{\mu_0}{4\pi}\int\vec{\nabla}.\left(\frac{\vec{J}\times\vec{R}}{R^3}\right)d\tau$

$$=\frac{\mu_0}{4\pi}\int\left[\frac{\vec{R}}{R^3}\left(\vec{\nabla}\times\vec{J}(r')-\vec{J}\left(\vec{\nabla}\times\frac{\vec{R}}{R^3}\right)d\tau=0\right.\right.$$

(so $\vec{\nabla}\times\frac{\vec{R}}{R^3}=0,$ and $\vec{\nabla}$ acts only on field co-ordinates)

10. Solution

The force on the dipole m_2 due to m_1 is given by

$$\vec{F}=\left(\vec{m}_2.\vec{\nabla}\right)\vec{B}_1,$$

where $\vec{B}_1=\frac{\mu_0}{4\pi}\left[-\frac{\vec{m}_1}{r^3}+3\frac{\left(\vec{m}_1.\vec{r}\right)}{r^5}\vec{r}\right]$

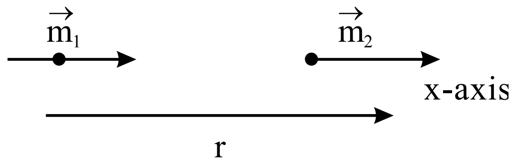
$$r=x\hat{i}+y\hat{j}+z\hat{k}=r\hat{i}$$

$m_1=m_1\hat{i}$, and $m_2=m_2\hat{i}$

Sol^n 10.

$$\vec{F}=\left(m_2\frac{\partial}{\partial x}\right)\left[\frac{\mu_0}{4\pi r^3}\left(-m_1+3m_1\right)\right]\hat{i}$$

$$= \hat{i}\,\frac{\mu_0 m_1 m_2}{2\pi}\frac{\partial}{\partial x}\left(1/x^3\right)$$



$$= -\hat{i}\,\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{x^4} - = -\hat{i}\,\frac{\mu_0}{4\pi}\frac{6 m_1 m_2}{x^4}$$

which is the force of allraction along the line joining the dipole.

Solution 11.

Magnetic field is uniform inside a role roid, of n turms, and carying current I is equal to

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & r \le R \\ = 0 & r > R \end{cases}$$

$$\oint \vec{A}.d\vec{l} = \int_S \left(\vec{\nabla}\times\vec{A}\right).d\vec{a} = \int \vec{B}.d\vec{a}$$

Now considering a closed loop of radius r insicle the solenoid.

$\left|\vec{A}\right|.2\pi r = \mu_0 n I\left(\pi r^2\right)$ when $r \le R$

$$\therefore\ \vec{A} = \frac{\mu_0 n I}{2} r\hat{\Phi}$$

Direction of $\vec{A}$ along the direction of I.

Now considering a closed circular loop of radius r.

Outside the solemoid,

$$\left|\vec{A}\right| = 2\pi r = \left|\vec{B}\right|\cdot\pi R^2$$

$\therefore\ \left|\vec{A}\right| = \dfrac{\mu_0 n I}{2r} R^2\hat{\Phi}$ (Since the field extends up to r = a)

Hence $\vec{A}\begin{cases} \mu_0 I n\hat{\Phi} & r \le R \\ \dfrac{\mu_0 I n}{2r} R^2\hat{\Phi} & r > R \end{cases}$

3.12 Problems and Solutions

1. A proton and a deuteron have equal kinetic energies. Compare the radii of their paths when a magnetic field is applied normal to their orbits.

2. A particle of charge q and mass m is projected with a velocity v perpendicular to a uniform megnetic field of field intensity B. Find the angle of deviation θ when

i) d < r (radius of rotation of charge) ii) d = r iii) d < r

3. A square loop of side 'a' carries a current in anti-clockwise direction when viewed normally. Calculate the magnetic field at the center of the coil.

4. An electron is rotating of charge e is rotating frequency n around the nucleus in a circular orbit of radius r. Find the magnetic field B at the position of necleus. $r = 5.1 \times 10^{-11}$ m and $n = 6.8 \times 10^{15}$ Hz.

5. A particle with charge q is projected successively along x-axis and y-axis with same velocity $\vec{v}$. The force on the particle in these situations is given by $vqB\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}\hat{k}\right]$ and $vqB\left[-\frac{1}{2}\hat{i}\right]$ respectively. Find the direction of magnetic induction $\vec{B}$.

6. Show that the vector potential for a uniform magnetic field $\vec{B}$ along z-direction is given by $\vec{A} = -\frac{1}{2}\left(\vec{r}\times\vec{B}\right)$

7. The fig (1) and fig. (2) shows two circuits with loop radius 2r and r but with different orientation of loop. Find the ratio of magnetic field produced at the centres o_1 and o_2.

8. The magnetic vector potential in a region is defined by c $\vec{A} = e^{-y}\sin x\hat{k}$. An infinitely long coductor, having a cross section area, a = 5 mm^2 and carrying a dc current I = 5A in the y-direction, passes through the region as is fig (3)

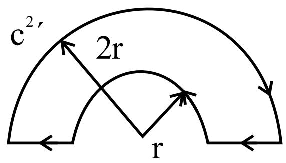


Fig. 1

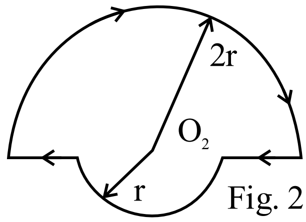
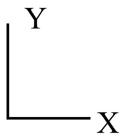


Fig. 2

Determine the expression for (a) $\vec{B}$ and (b) the force density $\vec{f}$ exerted on the conductor.

9. The fig (4) shows a conducting wire carrying current i placed perpendicular to a uniform magnetic field $\vec{B}$ acting verfically into the plane of paper. AB and CD have length l and circular are BC has radius l. Find the force on wire.



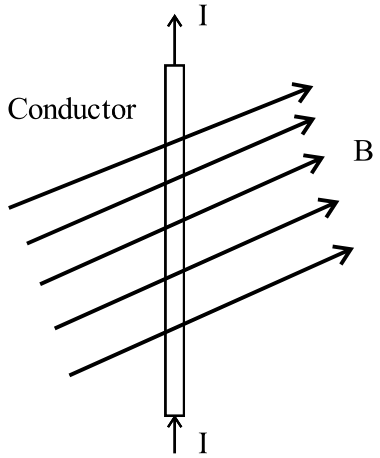


Fig. 3

10. The fig (6) shows a very very long wire carrying current i, with its middle portion bent in the form of a circular are of radius r as shown in fig (6). Find the magnetic induction $\vec{B}$ at the centre O of arc.

11. A steady current I flows down a long cylindrical conductor of radius a. The current density at a distance r from the axis of the conductor is proportional to r. Calculate the magnetic field both, inside and out side the wire as a functor of r.

12. (a) A particle of mass m and charge q is rotating in a circle of radius a with an angular velocity w. Show that the ratio of its magnetic moment to mechanical moment angular momentum is q/2m

(b) If the magntiude of the angular momentum of a electron rotating in a circular orbit is L find its magnetic moment.

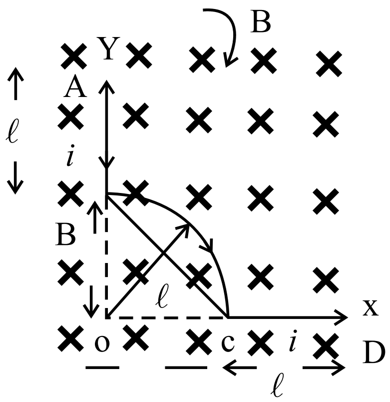


Fig. 5

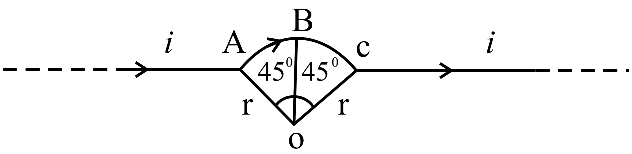


Fig. 6

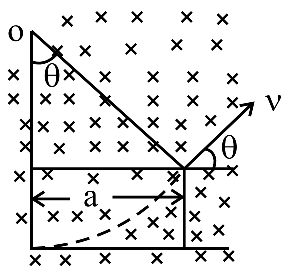


Fig. 3.32

Solution of numerical problems

1. Let v_p and v_d be the veolcities of proton and deuteron. m be the mass of proton, then the mass of deuteron will be 2m. Their respective radius be r_p and r_d. Then in a magentic field B normal to their path,

$Bqv_p = mv_p^2/r_p$ ..... (3.12.1)

$Bqv_d = 2mv_d^2/r_p$ ..... (3.12.3)

So $r_p/r_p = v_p/2V_d$ ..... (3.12.3)

By the condition $mv_p^2/2 = 2mv_d^2/2$, so $v_p/v_d = 1/\sqrt{2}$. Comparing with equation 3.12.3 we have $r_d = \sqrt{2}r_p$

2. Here r = mv\qB. So from the fig (3.32) sinθ = d/r = Bqd/mv. So,

i) if r > d, θ is acute and the particle will migrate in next medium.

ii) if r = d, θ = π/2 and the particle will graze the surface of separation.

iii) if r < d, the particle will describe a half circle and return back to the previous medium parallel to itself.

3. The fig (3.33) shows a square loop carrying current

i. The magnetic induction at centre o due to arm DA

$$B_{DA} = \frac{\mu_0}{4\pi}\frac{i}{d/2}\left(\sin 45^\circ + \sin 45^\circ\right)$$

$= \frac{\mu_0}{4\pi} 2\sqrt{2}\frac{i}{d}$ perpendicular to ABCD and directed normally up the plane of paper.

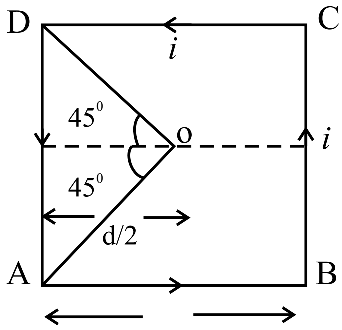


Fig. 3.33

So the total magnetic induction

$B = B_{ABCD} = 4\frac{\mu_0}{4\pi} 2\sqrt{2}\frac{i}{d} = \frac{2\sqrt{2}\mu_0 i}{\pi d}$ acting normally up the plane of paper.

4. Here $i = en = 1.6\times10^{-19}\times6.8\times10^{15} = 10.22\times10^{-4} A$

$$B = \frac{4\pi\times10^{-7}\times10.22\times10^{-4}}{2\times5.1\times10^{-11}} = 14.4T \text{ ... ...3.23}$$

5. We know magnetic force $\vec{F} = q\vec{v} \times \vec{B} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_2 \\ B_x & B_y & B_z \end{vmatrix}$

Now for first condition magnetic field should be in y-z plane So, $\vec{F} = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & o & o \\ o & B_y & B_z \end{vmatrix}$

That gives qv By $= qv\dfrac{\sqrt{3}}{2}B$ (Comparing the z-compones of force)

So By $= \dfrac{\sqrt{3}}{2}$ B.... (3.12.4)

Imposing the second condition similarly we have B_2 = B/2 .... (3.12.5)

So $\vec{B} = \dfrac{B}{2}\left(\sqrt{3}\hat{j} + \hat{k}\right)$

Let θ be the angle with y-axis.

$$\therefore \vec{B}.\hat{j} = \sqrt{\left(\frac{\sqrt{3}}{2}B\right)^2 + \frac{B^2}{4}} \cos\theta = \frac{\sqrt{3}}{2}B$$

$$\therefore \cos\theta = \frac{\sqrt{3}}{2} \quad \therefore \theta = 30^{\circ}$$

6. Here $\vec{B} = B\hat{k}$, So $\nabla \times \vec{A} \begin{pmatrix} i & j & k \\ \dfrac{\partial A_x}{\partial x} & \dfrac{\partial A_y}{\partial y} & \dfrac{\partial A_z}{\partial z} \\ \mathrm{Ax} & \mathrm{Ay} & \mathrm{Az} \end{pmatrix}$ *gives*

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

& $B = \frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}$

So $\vec{A}$ will be either propertional to $\vec{B}$ and a linear function of r. We can take two choice 1. $\vec{A} = C_1 B\vec{r}$ 2. $\vec{A} = C_2\left(\vec{B} \times \vec{r}\right)$

As for first choice $\vec{B} = \nabla \times \vec{A} = 0,$ so it is the The second choice $\vec{A} = C_2 \begin{vmatrix} i & j & k \\ 0 & 0 & B \\ x & y & z \end{vmatrix}$

yields

$A_x = -C_2 yB$ & $A_z = 0$

$Ay = C_2 Bx$

Now $\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -c_2 yB & C_2 Bx & o \end{vmatrix} = \hat{K}B$

Or $B = C_2B + C_2B = 2BC_2$ Or, $C_2 = 1/2$

Thus $\vec{A} = -\frac{1}{2}\left(\vec{r} \times \vec{B}\right)$ from 2nd choice

7. In fig (1.21), The magnetic is O_1

$\vec{B}_1 = \frac{\mu_0}{4\pi} i\left(\frac{\pi}{r} - \frac{\pi}{2r}\right) = \frac{\mu_0}{4\pi} i \frac{\pi}{2r}$ up the plane of paper.

In fig (2), The magnetic induction at O_2

$\vec{B}_2 = \frac{\mu_0}{4\pi} i\left(\frac{\pi}{r} + \frac{\pi}{2r}\right) = \frac{\mu_0}{4\pi} i \frac{\pi}{r}\frac{3}{2}$ up the plane of paper.

$\therefore \left|\vec{B}_2\right| : \left|\vec{B}_1\right| = 3.$

8. $\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & e^{-y} \sin x \end{vmatrix}$

Or, $\vec{B} = -e^{-y} \sin x\hat{i} - e^{-y} \cos x\hat{j}$

$= -e^{-y}\left(\hat{i} \sin x + \hat{y} \cos x\right)$

For exerted on a elemental length $d\vec{l}$ on conductor

$\vec{F} = I\left(d\vec{l} \times \vec{B}\right) = I\hat{j}dl \times \vec{B}$

$= Idl\hat{j} \times -\left(\hat{i} \sin x + \hat{j} \cos x\right)e^{-y}$

$= Idle^{-y} \sin x\hat{k} = 5e^{-y} \sin x\hat{k}\left(dl\right)$

$\therefore$ force density $\frac{\vec{F}}{dl} = 5e^{-y} \sin x\hat{k}\, N/m$

9. We set x & y axis as in figs

The force on arm AB, $\vec{F}_{AB} = \frac{\mu_0}{4\pi} ilB\hat{i}$

The force on arm CD, $\vec{F}_{CD} = \frac{\mu_0}{4\pi} ilB\hat{j}$

The force on arm BC, $\vec{F}_{BC} = \frac{\mu_0}{4\pi} i\sqrt{2}\ \&\ B\frac{\left(\hat{i} + \hat{j}\right)}{\sqrt{2}}$

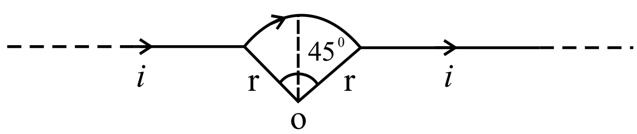


Fig. 3.32

$$= \frac{\mu_0}{4\pi} ilB\left(\hat{i} + \hat{j}\right)$$

So the total force $\vec{F} = \frac{2\mu_0}{4\pi} ilB\left(\hat{i} + \hat{j}\right)$

10. Magnetic induction at O due to straight portions of wire

$$\vec{B}_{stright} = 2\left[\frac{\mu_0}{4\pi} \frac{i}{r \sin 45^o}\left(\sin 90^o - \sin 45^o\right)\right]$$

$= \frac{\mu_0}{4\pi} \frac{i2}{r}\left(\sqrt{2} - 1\right)$ acting normally into the plane of paper.

$\vec{B}_{arc} = \frac{\mu_0}{4\pi} \frac{i}{r} \pi/2$ acting normally into the plane of paper.

Total magnetic induction $\vec{B} = \vec{B}_{straight} + \vec{B}_{arc} = \frac{\mu_0}{4\pi} \frac{i}{r}\left[\frac{\pi}{2} + 2\left(\sqrt{2} - 1\right)\right]$ acting normally into the plane of paper.

11. Solution

Considuring an Amperian loop in the form of a circle of radius r (r > a) with its axis on the axis of the cylinder, B is is tangential to the loop be cause of symmetry and constant over it

$$\oint \vec{B}.d\vec{l} = \mu_0 I_{encl}$$

Or, $B.2\pi r = \mu_0 \int_0^0 2\pi r dr.J\left(r\right)$

$$= 2\pi\mu_0 k \int_0^r r^2 dr = 2\pi\mu_0 k \frac{r^3}{3}$$

where k is portionality constatin J = Kr

$$\therefore B = \mu_0 k \frac{r^2}{3} \ for \ r \le a.$$

For any external point r > a, I_{end} = I and then, $B 2\pi r = \mu_0 I$

Or, $B = \dfrac{\mu_0 I}{2\pi r} \; for \; r \geq a$

Now the total current $I = \int_0^a 2\pi r dr . J(r) = 2\pi k \int_0^a r^2 dr$

$= 2\pi k \dfrac{a^3}{3}$ or, $k = \dfrac{3I}{2\pi a^3}$

Thus $B = \dfrac{\mu_0 I r^2}{2\pi a^3} \; for \; r \leq a$

a solution

(12.b) current flow due to charge q rotating in a circular orbit is

$$I = \frac{q}{T} = \frac{\frac{q}{2\pi}}{2\pi\nu} = \frac{qa}{2\pi}$$

Magneitc moment = current × area of the loop $M = I\pi a^2$

$$M = \frac{q\omega}{2\pi} \frac{\pi a^2 m}{m}$$

$$= \frac{q}{2m} mwa^2$$

$= \dfrac{q}{2m} \times$ Angular momentum.

$$\frac{\text{Magnetic moment}}{\text{A ngular momentum}} = \frac{q}{2m}$$

From above, taking q = e as charge of electron. Then

$$\frac{\text{Magnetic moment}}{L} = \frac{e}{2m} \Rightarrow M = \frac{eL}{2m}$$

Unit 4 ❑ Magnetic Properties of Matter

Structure

4.1 Objectives

You will know from this unit—

- What is magnetisation and its measurement (M)
- Behaviour of closed circulating current and its relation to non-uniform magnetisation.

- Relation between auxiliary magnetic field $\vec{H}$, magnetic induction vector $\vec{B}$, and intensity of magnetisation $\vec{M}$.
- Hysteresis loss for ferromagnetic material and its importance.
- Classification of magnetic material according to their property mainly pare dia and ferromagnetic material.

4.2 Introduction

In our earlier unit we have studied the primary properties of magnetism. Apart from the directive, and altractive/repulssive properties, the fundamental property of a magnetic field is that its flux through any closed surface vanishes. Mathematically it is expressed as $\vec{\nabla}.\vec{B}=0$ i.e. these field lines close on themselves.

The most common source of magnetic fields is the electric current loop. It may be an electric current in a circular conductor or the motion of an orbiting electron in an atom. Associated with both types of current loops is a magnetic dipole moment, the value of which is iA, the product of current (i) and area of the loop (A). Besides these, electrons protons, and neutrons in atoms have a magnetic dipole moments for their intrinsic spin property.

At present, we will study more about properties of magnetism, and intensity of magnetisation. The nature of circulating current related to non-uniform magnetisation and the relation between current density and intensity of magnetisation will be studied in detail here.

A simple relation between auxiliary magnetic field ($\vec{H}$) and magnetic induction ($\vec{B}$). their relation will be established here. Magnetic material is classified into three main category—para, dia and ferromagnetism. Their general properties are included here particularly, ferromagnetic material with their hysteresis property are relevant in fabricating temporary or permanent magnet. Ferromagnetic material is of immense use in industry, i.e. in transformer design.

4.3 Magnetisation and its Measurement

In an atom, electron revoles around the nucleus in different orbits, so we can say that each orbit is closed electrical circuit, which acts as a magnetic dipole. Magnetism of this closed electrically orbital circuit or magnetic dipole can be expressed in terms of Magnetic polarisation. It is the active current flow and a is it surface area, then magnetic dipole polarisation of each orbit,

$$m = Ia \qquad \text{......................(4.3.1)}$$

Normally, these orbits or dipoles are randomly distributed, so in effect, resultant magnetic effect of these get neutralised. They try to orient themselves in order under the infulence of external magnetic field. So the material retains magnetic properties. Total number of dipoles oriented along the externally applied field is defined as the intensity of magnetisation $\vec{M}$ If V is the volume and total number of dipoles Σm_i in it, then magnetisation M is given by

$$M = \frac{\Sigma m_i}{V} \qquad \text{.... (4.3.2)}$$

where m_i is the ith magnet dipole value.

4.3.1 Equivalency between Magnetic Circuit and Electrical Circuit :

Let us take a piece of magnetic material. This piece can be imagined to be assembly of small mesh structures. As current is the source of magnetisation, so the magnetic behaviour of every micro mesh can be considered due to the flow of current in one direction. This is portrayed in Fig 4.l. This current flow is same for every network for uniform magnetisation.

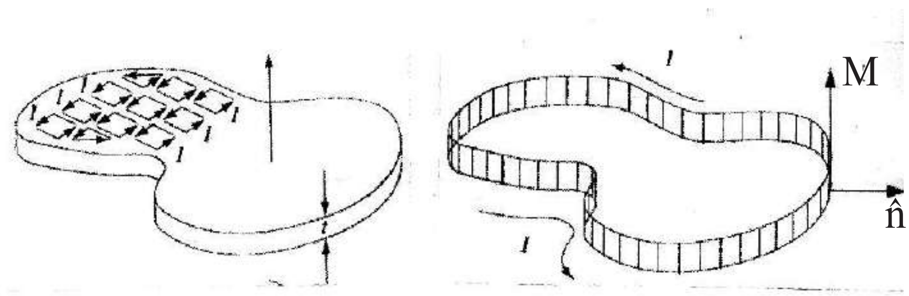


Fig. 4.1

It is clear from the figure that current in adjacent orbital circuit or mesh is equal and in opposite direction, is neutralised by each other, only the current flow left out in external boundary or periphery of the collected mesh does not vanish and remain active. It can be concluded that a magnetic material is an arrangement of equally structured numbers of orbitally current curcuit mesh work for uniform magnetisation. This active current around the periphery is called circulating current. The characteristics feature of this current is that it is not due to freely moving electrons. It is the current produced by electrons revolving in atoms of different structural configuration of magnetic material. So this current is called bound current. A relation ship between magnetisation and bound current can be derived as follows.

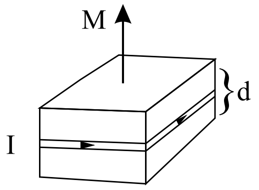


Fig. 4.2

In Fig. 4.2 a small sample of magnetic material is displayed, whose area is 'a' and its breath is d. M is intensity of magnetisation and m its dipole moment, then we get,

$$m = Mad \qquad ....(4.3.1.3)$$

As the magnetic material is like an electrical circuit of equally spaced block, and I is the governing current then magnetic dipole moment will be,

$$m = Ia \qquad ....(4.3.1.4)$$

coparing equations (4.3.3) and (4.3.4) we get,

$$M = \frac{I}{d} = K \qquad ....(4.3.1.5)$$

K is known as surface current density. It is clear from equation (4.3.5) that magnetisation intensity and surface current density are identical. The direction of the current on each surface is given by

$$\vec{K} = \vec{M} \times \hat{n} \qquad ....(4.3.1.6)$$

This equation is very important. Here $\hat{n}$ is the unit surface vector. Now surface vector $\vec{K}$ directed externally outward. $\vec{M}$ and $\hat{n}$, being parallel to each other, current flow in upper and lower surface have no existence.

4.3.2 Relation between Magnetisation and Current Density in Non Uniform Magnetisation :

Magnetisation current is active in adjacent boundary of mesh block of magnetic material in non-uniform magnetisation. Two adjacent block of magnetic material is shown in Fig. 4.3. Magnetisation is not uniform everywhere, so intensity of magnetisation is different in two blocks. Let $M_Z(y)$ and $M_Z(y+\Delta y)$ are magnetic intensity of two blocks, respectively. Equation (4.3.5) gives the current density. So the current flow in each block will be different. Arrow sign indicates the direction of flow of current. Let $I_x(1)$ and $I_x(2)$

be the current flow of the blocks, respectively. From equation (4.3.5) and Fig. 4.3, We get,

$$I_x(1) = M_z(y)\Delta z$$

and $$I_x(2) = M_z(y + \Delta y)\Delta z$$

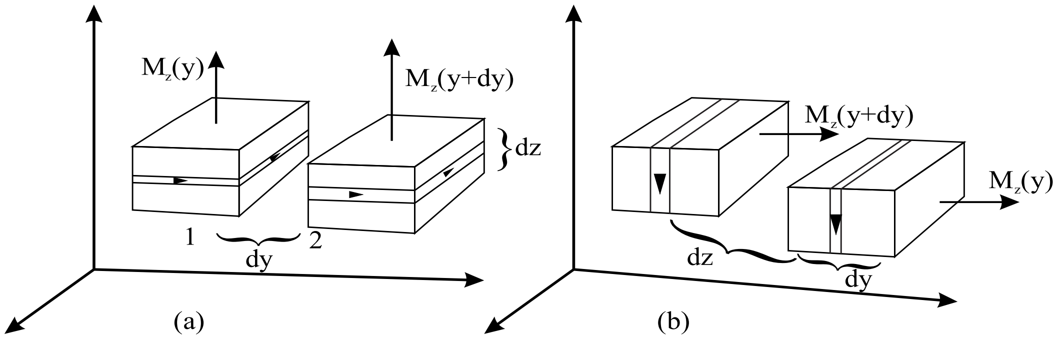


Fig. 4.3

At the junction of the two block $I_x(1)$ is –ve along x-axis and $I_x(2)$ is tve along X-axis and active. As $I_x(2)$ is grea ter than $I_x(1)$ so we can understand that current will be more active along positive X-axis. Remainder of the current flow will be

$$\Delta I_x = I_x(2) - I_x(1)$$

or, $$\Delta I_x = [M_z(y+\Delta y) - M_z(y)]\Delta z \qquad \text{......(4.3.7)}$$

Now, $M_z(y+\Delta y) = M_z(y) + \dfrac{\partial M_z}{\partial y}\Delta y +$ .... other negligible terms

So, $$\Delta I_x = \frac{\partial M_z}{\partial y}\Delta y \qquad \text{........(4.3.8)}$$

J is the current density per unit area and it's direction along per pendicular to the area, so current density due to unequal magnetisation along y-axis will be

$$(J_m)_{x^1} = \frac{\Delta I_x}{\Delta y \Delta z} = \frac{\partial Mz}{\partial y} \qquad \text{.......(4.3.9)}$$

This current density is due magnetisation, (that is why m is used as subscription).

The reasons given above is responsible for the origin of $(J_m)_{x1}$. It is clear from Fig. 4.3.(b) that residual x component of current will be due to the variation in magnetisation along z-axis and will remain active. Hence

$$(J_m)_{x2} = \left[\frac{M_y(z+\Delta z) - M_y(z)}{\Delta y \Delta z}\right]\Delta y$$

$$= \frac{\partial My}{\partial z} \qquad \text{........(4.3.10)}$$

So the total current density at any point due to uneven magnetisation will be

$$\left(\overrightarrow{J_m}\right)_x = \left(\overrightarrow{J_m}\right)_{x1} + \left(\overrightarrow{J_m}\right)_{x2}$$

$$= \frac{\partial \overrightarrow{M_z}}{\partial y} - \frac{\partial \overrightarrow{M_y}}{\partial z} \qquad \text{........(4.3.11)}$$

$$= (\vec{\nabla} \times \vec{M})_x$$

i.e. $\left(\overrightarrow{J_m}\right)_x$ is x component of $(\vec{\nabla} \times \vec{M})_x$

In this way we can derive the flow of current along, y and z axis, so the resultant current density will be

$$\overrightarrow{J_m} = \vec{\nabla} \times \overrightarrow{M} \qquad \text{........(4.3.12)}$$

Equation (4.3.12) is the relation between current density and intensity of magnetisation. $\vec{\nabla} \times \vec{M} = 0$, for uniform magnetic field or $J_m = 0$, the current flow remain active only along the of periphery. There will be no influence inside the magnetic material.

4.3.3 Alternative Method to Find $\vec{\nabla} \times \vec{J} = \vec{M}$

To find a quantitative relation between $\vec{M}$ and $\vec{J}$, let us consider magnetic vector potential due to a magnetised body as shown Fig. 4.4. The vector potential due to a single current loop of magnetic moment $\vec{m}$ is given by

$$\vec{A}(r) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2} \qquad \text{...........(4.3.13)}$$

where $\vec{r}$ is a radius vector from the loop to the point of observation. In a magnetised object, each volume element dz carries a dipole moment $\vec{M}$dz, so the total vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\overrightarrow{M}(\vec{r}) \times \widehat{r''}}{r''^2} d\tau \qquad \text{......(4.3.14)}$$

From vector algebra,

$$\vec{\nabla}\frac{1}{r''} = \frac{\hat{r}''}{r''^2}$$

with this

$$\vec{A}(\vec{r}) = \int\left[\overrightarrow{M}(\vec{r}) \times \left(\nabla' \frac{1}{r}\right) d\tau\right]$$

Integrating by parts

$$\vec{A}(\vec{r}) = \frac{M_0}{4\pi}\left[\int \frac{1}{r}\left(\vec{\nabla}' \times \overrightarrow{M}(\vec{r}')\right) d\tau - \int \nabla' \times \left\{\frac{\overrightarrow{M}(\vec{r}')}{r''} d\tau\right\}\right.$$

$$\text{or,} \quad \overrightarrow{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{r''}\left[\vec{\nabla}' \times \overrightarrow{M}(\vec{r}') d\tau\right] + \frac{\mu_0}{4\pi}\int \frac{1}{r''}\left[\overrightarrow{M}(\vec{r}') da'\right] \qquad \text{.......(4.3.15)}$$

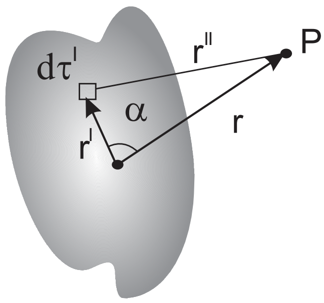


Fig. 4.4

The first term looks just like the potential of a volume current.

$$\vec{J} = \vec{\nabla} \times \vec{M} \qquad \text{......(4.3.16)}$$

while the second term looks like the potential of a surface current

$$\vec{K} = \vec{M} \times \hat{n} \qquad \text{.......(4.3.17)}$$

where $\hat{n}$ is the unit normal vector, with this definition

$$A(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')d\tau'}{r''} + \frac{\mu_0}{4\pi}\oint \frac{\vec{K}(r')da'}{r} \qquad ....(4.3.18)$$

Equation (4.3.18) shows that potential of magnetised object is the same as would be produced by a volume current $\vec{J} = \vec{\nabla} \times \vec{M}$ throughout the magnetic material, plus a surface current $\vec{K} = \vec{M} \times \hat{n}$ on the boundary.

4.4 Auxiliary Magnetic Field ($\vec{H}$)

Now we place a magnetic material inside a solenoidal coil and a current I_f is flown across it from a battery. If total current density is $\vec{J}$, then

$$\vec{J} = \vec{J}_f + \vec{J}_m$$

Here $\vec{J}_f$ and $\vec{J}_m$ are free current density and bound current density. From Ampere's circuital law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\vec{J}_f + \vec{J}_m) \qquad ....(4.4.1)$$

As $\quad \vec{J}_m = \vec{\nabla} \times \vec{M}$

So, $\quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \vec{\nabla} \times \vec{M}$

or, $\quad \vec{\nabla} \times \vec{B} / \mu_0 - \vec{\nabla} \times \vec{M} = \vec{J}_f$

or, $$\vec{\nabla} \times \left[\frac{\vec{B}}{\mu_0} - M\right] = \vec{J}_f \qquad ....(4.4.2)$$

is denoted as $\vec{H}$, equation (4.4.2)

becomes $\vec{\nabla} \times \vec{H} = \vec{J}_f \qquad ....(4.4.3)$

$\vec{H}$ is known as auxiliary magnetic field or magnetisation. In reality $\vec{H}$ is very important as it is directly related to the current flow from battery. If we study equation (4.4.1) and (4.4.3), we can conclude that magnetic induction vector $\vec{B}$ is related to total current flow, but cannot be measurable easily otherwise $\vec{H}$ can be measured easily as it is related free flow of current.

Applying stoke's theorem in equation (4.4.3) we get

$$\int\left(\vec{\nabla}\times\vec{H}\right).\hat{n}ds = \oint \vec{H}.\vec{dl} = \int \vec{J}_f.\hat{n}ds = J_f \qquad \text{........(4.4.4)}$$

Equation (4.4.4) states that an integration of $\vec{H}$ around a closed loop is linked with free current flow from EMF/other sources. This equation is frequently used to find $\vec{H}$.

We understand from the above analysis that application of auxiliary magnetic field $\vec{H}$ in material initiates the evolution of magnetic field, which is known as magnetic induction vector $\vec{B}$.

Now, $\dfrac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$

or, $\vec{B} = \mu_0\,(\vec{H} + \vec{M})$ ....(4.4.5)

Here μ_0 is the permeability of free space. Equation (4.4.5) is the relation between $\vec{B}$, $\vec{H}$ and $\vec{M}$.

4.5 Magnetic Permeability and Susceptibility

Magnetic permeability of a material is the ability of a material to support the formation of a magnetic field inside itself. So it is known as degree of magnetisation standard unit of magnetic permeability is Hm^{-1}.

The magnetic permeability is a relative measurement that it is taken with respect to the magnetic permeability of vacuum. A diamagnetic material has a relative permeability less than 1, Where as a paramagnetic material has a value slightly greater than one which means that when a paramagnetic material is placed in external magnetic field, it becomes slightly magnetised. But a ferromagnetic materials have relative permeability.

Magnetic susceptibility is the measure of magnetic properties of material which indicates whether the material is attracted or repell from external field. This is quantitative measurement of the magnetic properties. It is denoted as χ_m. For a isotropic linear magnetic material.

$$\vec{M} = \chi_m\vec{H} \qquad \text{............ (4.5.1)}$$

χ_m is a dimensionless quantity the values of χ_m for common para magnetic and diamagnetic materials are given below.

Table 4.1

Paramagnetism and Diamagnetism: Magnetic Susceptibilities

Paramagnetic substance	χ_m	**Diamagnetic substance**	χ_m
Aluminum	2.3×10^{-5}	Copper	-9.8×10^{-6}
Calcium	1.9×10^{-5}	Diamond	-2.2×10^{-5}
Magnesium	1.2×10^{-5}	Gold	-3.6×10^{-5}
Oxygen (STP)	2.1×10^{-6}	Lead	-1.7×10^{-5}
Platinum	2.9×10^{-4}	Nitrogen (STP)	-5.0×10^{-9}
Tungsten	6.8×10^{-5}	Silicon	-4.2×10^{-6}

Now we can estalish a simple relation between $\vec{B}$ and $\vec{H}$ from equations (4.4.5) and (4.5.1)

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (\vec{H} + \chi_m H)$$

$$= \mu_0 \vec{H}(1+\chi_m) \qquad \text{........(4.5.2)}$$

or, $$\vec{B} = \mu\vec{H} \qquad \text{.........(4.5.3)}$$

Here $\mu = \mu_0 (1 + \chi_m)$ ....(4.5.4)

where μ is the permeability of the medium. Also we get relative permeability, as

$$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m \qquad \text{........(4.4.5)}$$

μ_r is a dimensionless quantity. In vacuum, $\chi_m = 0$, $\mu_r = 1$.

4.6 Classification of Magnetic Materials

Magnetic materials can be classified according to the behaviour of magnetic moments of electron of an atom react to applied magnetic field diamagnetic, paramagnetic and ferromagnetic materials.

Diamagnetic Materials : They are weakly magnetised in a direction opposite to the applied magnetic field.

Examples are hydrogen, nitrogen, gold, silver, copper, antimony etc. Its behaviour with applied field and temperature are shown in Fig. 4.5

Explanation : Dia magnetic substance are composed of atoms that have no net magnetic moments. When it is placed in an external magnetic field, the substance as a whole acquires net magnetic moment in a direction opposite to the applied field. They do not have unpaired electrons.

Characteristics : 1. Diamagnetic is universal, all materials when exposed to external magnetic field, tend to develop magnetic moments opposite to the direction of the applied field.

2. No parmanent dipoles.

3. Relative permeability is less than one but positive.

4. Susceptibility is negative and small, independent of temperature. (Fig. 4.5)

5. Weak repulsion is its main features.

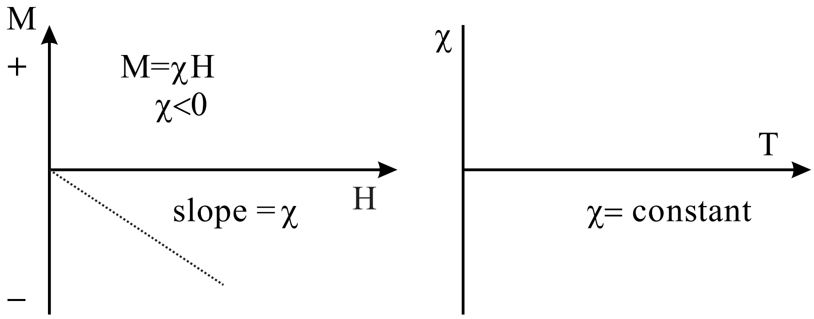


Fig. 4.5

Paramagnetic Materials : It is the phenomenon by which the orientations of the magnetic moments are mainly dependent on temperatrure and applied field. The number of orientations of orbital and spin magnetic moments be such that the vector sum of the magnetic moments is not zero. Resultant magnetic moment in each atom is not zero even in absence of field. Paramagnetic property vanishes in the absence of external field. Its behaviour with applied field and temperature are shown in Fig. 4.6. Examples are Aluminium, platinum, chromium, sodium, calcium, oxygen etc.

Paramagnetism Characteristics :

a. Paramagnetic materials have an unpaired electron in their valence shell

b. These unpaired electrons are in constant spinning motion.

c. This incessant spin of the electron form a dipole moment, they act as small magnets themsleves

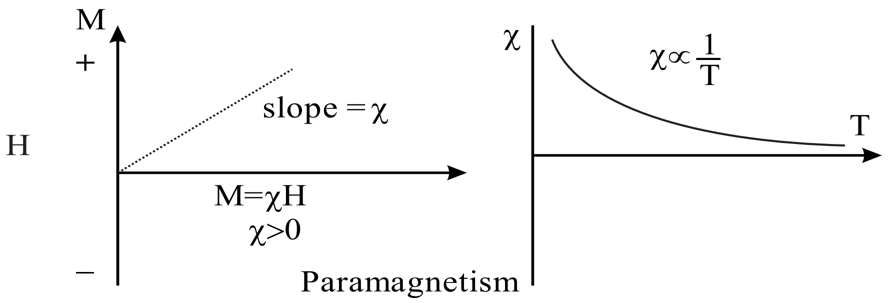
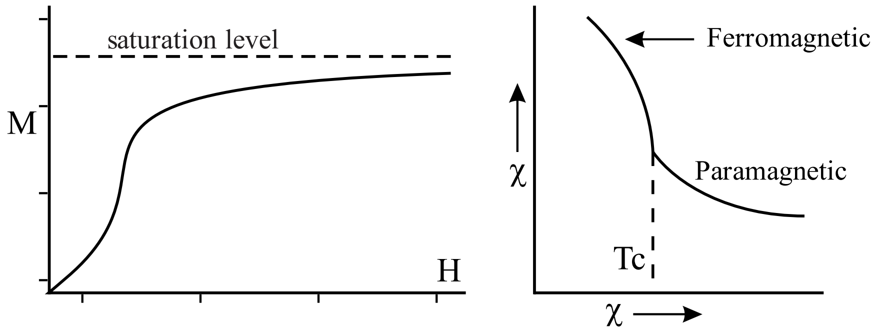


Fig. 4.6

d. However the dipoles are in random directions and donot interact with one another, so the total magnetic field created by paramagnetic material is zero.

e. In a magnetic field, the individual dipole moments of the atoms get alligned in a single direction, along the applied magnetic field. This produces a magnetic field in the direction of the applied field.

Ferromagnetic Materials : Ferromagnetism is the basic mechanism by which certain materials form permanent magnets or attracted to magnets. They are strongly magnetised in the same direction as that of applied field and retains its magnetic moment ever after removal of the applied field. Examples are Iron, cobalt, nickel. This property is due to the contribution of spin magnetic moment to the magnetic dipole moments is very large. It posseses strong magnetic properties due to the presence of magnetic domains. In these domains, large numbers of atomic moments (10^{12} to 10^{15} per unit volume) are alligned parallel, so that magnetic force within is strong. In unmagnetised state, the domain are nearly randomly organised, and the net magnetic field for the part as a whole is zero. The domains are oriented to produce a strong magnetic field under the influence of magnetising force.



Feromagnet ic Material Properties

Fig. 4.7 : Ferromagnetic Material Properties

Characteristics :

1. They have large and positive susceptibility.
2. They have strong tendency from weaker to the stronger parts of the non-uniform magnetic field.
3. $\vec{B} = \mu_0(\vec{H} + \vec{M})$. Magnetisation is not proportional to the applied field. They exhibit property called hysteresis.
4. Susceptibility depends on the temperature.
5. It retains their magnetic property even after the external field is removed.

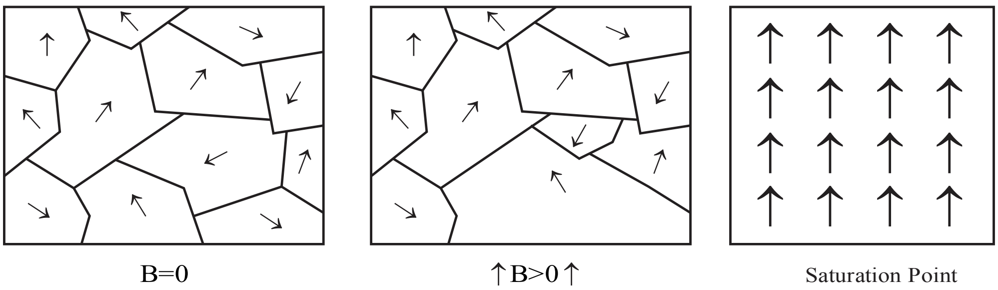


Fig. 4.8 : Allignment ferromagnetic domain with magnetic field

4.7 Relation between $\vec{B}$ and $\vec{H}$ of Material in Magnetic Field

We know that relation between magnetic induction vector $\vec{B}$ and auxiliary magnetic field vector $\vec{H}$ as $\vec{B} = \mu\vec{H}$. In Fig. 4.9, an experimental setup to find out the relation between

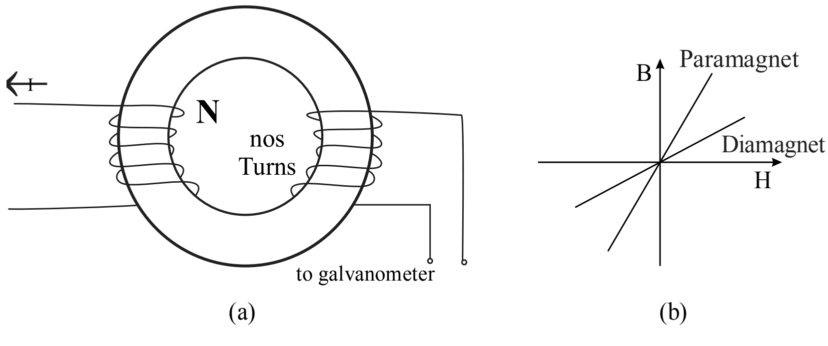


Fig. 4.9

$\vec{B}$ and $\vec{H}$. A magnetic material is introduced inside a torroid. Two coils have been wound on the torroid, one is primary coil, through which current is passed. Other is secondary coil, which has been connected to galvanometer. The magnetic field is produced by the current flow in the primary, which results in magnetic flux inside the torroid.

Magnetic flux is varied by the change in current flow. Induced emf can be measured. Thus $\vec{B}$ can be measured.

In this way, we can find the auxiliary magnetic field $\vec{H}$ and induced $\vec{B}$. Thus a graph can be plotted from this data. $\vec{B}$ and $\vec{H}$ graphs can be drawn for different material by placing it inside the torroid. A straight line graph is observed for paramagnetic and diamagnetic material. The gradient of the straight gives the susceptibility χ_m [Fig. 4.9(b)].

4.8 Hysteresis or Magnetisation Cycle

Magnetic properties of ferromagnetic material is different from other materials, which is clear from Fig. 4.10. $\vec{B}$ and $\vec{H}$ graph. Behaviour of $\vec{M}$ and $\vec{H}$ is not directly proportional and graph of $\vec{B}-\vec{H}$ is like a loop or cycle. Characteristic features of $\vec{B}-\vec{H}$ from the graph in Fig 4.10 are detailed below :

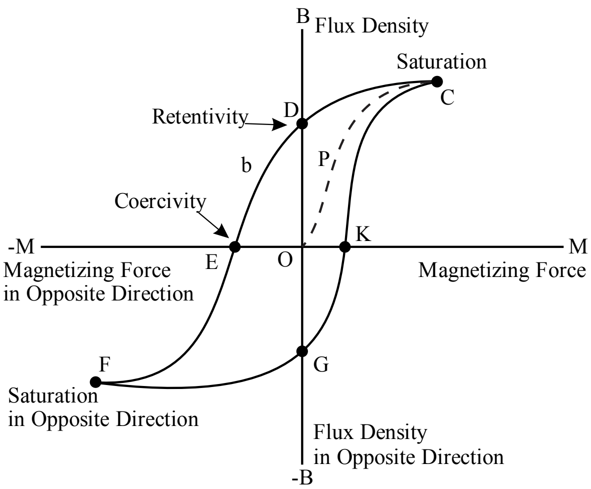


Fig. 4.10 : B – H Graph of ferromagnetic material

i) When I = 0, then H = 0, B = 0 and M = 0, so the point 'O' shows the unmagnetised state of the material.

Initially, we increase current slowly i.e. B increases as H increases. Magnetisation M has reached saturated state at the point C, i.e. H also is saturated. As $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, so B increases very little even if H is increased. OPC portion of the graph shows such variation.

ii) Now we decrease the current slowly after achieving saturated M at the point. We observe that B – H graph, instead of retracing the path OPC, it traverses along path CD i.e. B does not become zero value even when H = 0. Section CD portrays the residual magnetisation.

iii) Now if we reverse the flow of current and increase it slowly, H is also increased along the DEF portion of the graph. Residual magnetisation gets lost if the intensity of magnetic field is in the opposite direction along OE. The value of this applied magnetic field is called co-elective force. The point F describes the sate of saturated magnetisation in opposite direction.

iv) H is gradually increased from the point F in the reverse direction, and gets zero value. As a consequence we get the curve FGKC portion of the graph. Thus the material achieves previous value of magnetisation at the point C. The path CDEFGKC will be repeated again and again if I or H is changed repeatedly in forward and reverse direction, but the traversing path OPC of the graph is not traced at all. This closed path CDEFGKC is called magnetisation cycle or Hysteresis loop. At each step M or B lags behind the corresponding magnetising field H. It will be shown latter that the area of B – H loop equals to energy loss per unit volume per cycle of magnetisation.

4.8.1 Hysteresis Loss :

When a sample of ferromagnetic material is subjected to magnetisation and demagnetisation, in a hysteresis cycle, some amount of energy dissipated. As the magnetic domains allign with the magnetising field first in one way and then the other it produces mechanical stress and consequent heating. The energy spent during magnetisation is not totally recovered due to the irreversible changes in domain structure.

Let us take a typical domain of magnetic material of magnetic moment $\vec{m}$ which makes an angle θ with the field $\vec{H}$ at any instant of time when the magnetisation of the material is $\vec{M}$.

$\therefore$ The sum of the components of $\vec{m}$ perpendicular to $\vec{H}$ over unit volume is zero

$$\Sigma m \sin\theta = 0$$

Again, $M = \Sigma m\cos\theta$

$dM = -\Sigma m\sin\theta . d\theta$

Now the torque acting on the domain of moment $\vec{m}$ when it makes an angle θ with $\vec{H}$ is

$$|\tau| = \left|\vec{m} \times \vec{B}\right| = m\mu_0 H \sin\theta \qquad ....(4.8.1)$$

where $\vec{B} = \mu_0\vec{H}$ = magnetic induction due to $\vec{H}$.

The work done by the external source in alliging the domain in a unit volume through a additional angle is

$$dw = \Sigma m\mu_0 H \sin\theta d\theta = \mu_0 HdM \qquad ......4.8.2)$$

So the work done per unit volume in traversing the specimen through a complete cycle of magnetisation is

$$W = \mu_0 \oint HdM = \mu_0 \times \text{area enclosed by the M – H loop.} \qquad (4.8.3)$$

or, $W = \mu_0 \oint HdM = \oint HdB =$ area enclosed by the B–H loop.

4.8.2 Increase in Temperature Due to Hysteresis Loss :

Let A be the area of B – H loop and n is the number of hysteresis cycle, m is mass and ρ the density. ρ is the specific heat T is the temperature.

Energy lost per sec is = volume of the material × number of cycle × area of the loop

$$= \frac{m}{\rho} nAJ/s$$

Heat energy produced = $\frac{mnA}{J\rho}$ Calories/s.

By calore metric principle = $msT = \frac{mnA}{J\rho}$

$$\text{So } T = \frac{nA}{Js\rho} \ {}^{o}c$$

4.9 Importance of Hysteresis Loop

We can understand the nature of magnetic behaviour of a material by studying the structure of a hysteresis loop and identity its utility in manufacturing magnet. Hysteresis loop of soft iron and steel is shown in Fig 4.11 certain conclusion can be drawn are as follow :

(i) Retentivity of steel is more than solf iron.

(ii) Coercivity of steel is more than soft iron i.e. much greater coercive force is necessary to demagnetise steel-magnet.

(iii) Area of hysteresis loop of iron is much lesser than steel i.e. energy spent per cycle for soft iron is much less than steel.

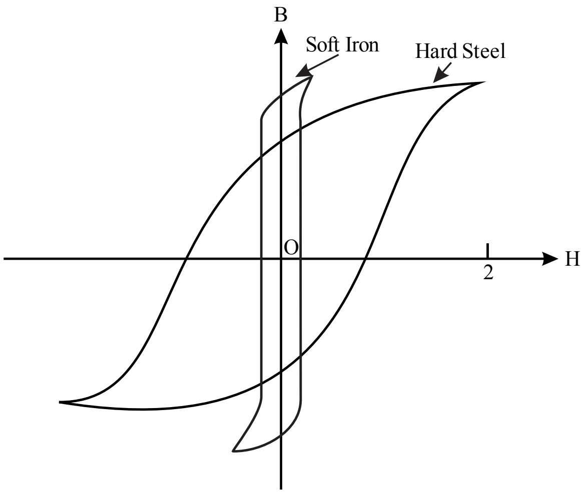


Fig. 4.11

So, we can understand why soft iron core is required in the manufacturing of electromagnet of transformer. Because transsient magnetism requires smaller area of loop and lesser coercive force. Larger coercivity is the necessity to have strong magnet. A strong magnet does not undergo a complete magnetic cycle. So energy lost due to hysteresis cycle in strong magnet, even though area is having much large area.

4.9.1 Demagnetisation of Magnetic Material

Ferromagnetic material retains some magnetism when it undergoes a hysteresis cycle. Magnetic transformation is unaltered even after with drawal of the applied magnetic field.

In order to demagnetise it, the magnet is placed in a gradually diminishing field and undergoes few hysteresis cycle. Area of the loop decreases gradually until it becomes zero. Thus the material has reached the state complete demagnetic state. (Fig. 4.12)

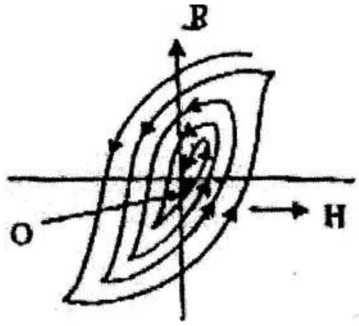


Fig. 4.12

4.10 Summary

1. Magnetisation of material is measured by $\vec{M}$. Numbers of magnetic dipole produced per unit volume, if material, is defined as magnetic moment.
2. Magnetic material behaves as composed of huge numbers of equal area circulating current loop, in uniform magnetisation. And the result at current flows only through peripheral region of the material.
3. Bound current exists in non-uniform magnetisation. Density of the current flow is given by $\vec{J}_m = \vec{\nabla} \times \vec{M}$. $\vec{\nabla} \times \vec{B} = \mu_0\vec{J} = m_0 (\vec{J}_f + \vec{J}_m)$ where $\vec{J}_f$ is the free current density sourced from battery/other sources and $\vec{J}_m$ is bound current density due magnetisation.
4. $\vec{\nabla} \times \vec{H} = \vec{J}_f$ and, $\oint \vec{H}.\vec{dl} = NI_f$ where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, the auxiliary magnetic field only related to the free current If. $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ for paramagnetic and diamagnetic material.
5. Hysteresis is the characteristics properties of feromagnet. Total energy spent in a hysteresis cycle is the area of the loop in SI.
6. Structure of the loop helps in identifying certain material with a specific purpose.

4.10 Review Questions and Answers

1. An electron (charge e mass m) revolves around the nucleus in a circular

orbit with radius r and velocity v. The electrostatic force provides the necessary centripetal force.

a) Calculate the current and magnetic dipole moment due to the orbital motion of the electron.

b) Write down the force equation when the electron is placed in a uniform magnetic field $\vec{B}$ perpendicular $\vec{v}$, how is the force equation modified?

Answer : a) Orbital current due to the electron.

$$I = \frac{e}{T} = \frac{e\omega}{2\pi} = \frac{ev}{2\pi r}$$, where T is time period, ω its augular velocity,

The magnetic dipole moment m = $I\pi r^2$

$$\frac{ev}{2\pi r}.\pi r^2 = \frac{evr}{2}$$

b) The force equation will be, $\frac{1}{4\pi \in_0} \frac{e^2}{r^2} = \frac{mv^2}{r}$

c) The electron will experience a force evB and its velocity rises from v to $\bar{v}$ in the presence of magnetic field.

The equation of motion will be—

$$\frac{1}{4\pi \in_0} \frac{e^2}{r^2} + e\bar{v}B = me\frac{\bar{v}^2}{r}$$

2. Determine the magnetisation current density due to non-uniform magnetisation current.

Answer : See article 4.3

3. If the magnitude of augular momentum of an electron rotating in a circular orbit is 'L' find the magnetic moment.

Answer : Orbital current due to electron, with time period T is

$$I = \frac{e}{T} = \frac{e}{\frac{2\pi r}{v}} = \frac{ev}{2\pi r}$$

So the magnetic moment

$$|\bar{m}| = I\pi r^2 = \frac{ev}{2\pi r}\pi r^2 = \frac{evr}{2}$$

We know, $v = \omega r$ and augular momentum $L = m\omega r^2$

$$|\vec{m}| = \frac{e\omega r^2}{2} = \frac{eL}{2m}$$

4. Using the concept of bound current density for non-uniform magnetisation, establish $\vec{\nabla} \times \vec{H} = \vec{J}_f$, where $\vec{J}_f$ is free current density.

Answer : See article 4.4.

5. Explain what you mean by free current and bound current in magnetisation of matter.

Answer : Free current is produced by the electric charges, like electron when they move. It produces Joule's heating effect. Bound current is produced by the orbital motion and spinning of electron in atom. It does not produces Joule's heating effect.

In magnetised matter atomic loops of current circuit are distributed at random. In uniform magnetisation, produced by the adjacent current loops cancel each other. Hence net effective current inside the material vanishes. Only we get some amount of surface current. In non-uniform magnetisation of matter, cancelletion will be partial and donot vanish. This residual current inside the maternal is called volume current. Thus we get formation of a current, which we call magnetisation current on bound current.

6. Discuss why soft iron is suitable for use as the core of transformer where as steel is preferred for making permanent magnet.

Answer : The core of a transformer is made of soft iron because it has high permeability so it provide complete linkage of magnetic flux of the primary coil to the secondary coil. Therefore it has high coercivity and low retentivity. Soft iron provides the best material for the core of a transformer as its permeability (μ) is very high. Its hysteresis curve is of small area and its coercivity low.

A permanent magnet requeres high retentivity and high coercivity. Steel magnet has this property and is able to resist loss of magnetism due to improper handling.

4.11 Problems and Solutions

1. An infinitely long cylinder of radius R carries a frozen-in magnetisation, parallel to the axis.

$$\vec{M} = kr\hat{z}$$

where K is a constant, and r is the distance from the axis (there is no free current here). (a) Find the bound current. (b) Find the magnetic field inside and outside. (c) Use Ampere's law to find $\vec{H}$ and $\vec{B}$.

Solution :

a) Given the magnetisation of the material along z-axis and is equal to

$$\vec{M} = k\vec{r}\hat{z}$$

The bound volume current is given $\vec{J} = \vec{\nabla} \times \vec{M}$

$$\vec{J}_b = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & kr \end{vmatrix}$$

$$= \frac{1}{r}\left[-\frac{\partial}{\partial r}(kr)\right]r\hat{\theta} = \frac{1}{r}\left[-\frac{\partial}{\partial r}(kr)\right]r\hat{\theta}$$

$$\therefore\ J_b = -k\hat{\theta}$$

The bound surface current is given by $-\vec{K}_b = \vec{M} \times \hat{n}$

$$\vec{K}_b = kr\ (\hat{z} \times \hat{r})$$

$$\therefore\ K_b = Kr\hat{\theta}$$

b) A sobenoidal field is produced due to bound current. The field outside the cylinder will be directed along the z-axis Applying Ampere's law we get,

$$\oint \vec{B}.\vec{dl} = -BL$$

The current intercepted by the Ampere's loop is given by

$$I_{in} = -K_b L + \int_r^R KL dr$$

$$= -(KLR + KL(R-r) = -KLr$$

Ampere's law can now be used to find the magnetic field.

$$\vec{B} = \frac{\mu_0 J_n}{-L}\hat{z} = \mu_0 Kr\hat{z}$$

(c) Now $\vec{\nabla}.\vec{M} = 0$, it implies Ampere's law uniquely defines $\vec{H}$. Now the $\vec{H}$ field is painting in z-direction. Using Ampere's law, in terms of the $\vec{H}$ field, we certainly conclude that for the Ampere's law

$$\oint \vec{H}.d\vec{l} = HL = J_{intercepted} = 0$$

Since there is no free current, which can be only true if $\vec{H} = 0$, which implies that

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} = 0$$

So the magnetic field $\vec{B}$ is given by

$$\vec{B} = \mu_0\vec{H}$$

Magnetisation outside the cylinder is zero and therefore magnetic field is zero

$$\vec{B} = 0$$

For the region inside the cylinder

$$M = kr\hat{z}$$

So internal magnetic field

$$\vec{B} = \mu_0 kr\hat{z}$$

which is identical to earlier solution

2. An iron rod (density 7.7×10^3Kgm^{-3} and specific heat 470 JKg^{-1}) is subjected to cycles of magnetisation having frequency 50 cycles. If the area of B–H loop of the specimen is 6×10^3 Jm^{-3}. Calculate the rise in temperature per min.

Solution :

Hysteresis area enclosed by the B–H loop

= Energy lost per unit volume per cycle

= 6×10^3Jm^{-3}

$\therefore$ Energy lost per min = $6\times10^3\times50\times60\times\dfrac{m}{7.7\times10^3}$

where mis the mass of the sample.

Let T be the rise in temperature, we get,

$$mST = m \times 470 \times 7 = 6 \times 10^3 \times 50 \times 60 \; \frac{m}{7.7\times10^3}$$

$$\text{or,} \quad T = \frac{6\times10^3\times50\times60}{470\times7.7\times10^3} = 4.97^0 C(\min^{-1})$$

3. Compute the intensity of magnetisation of the bar magnet whose mass, magnetic moment and density are 400g, 2 Am^2 and $8gCm^{-3}$, respectively.

Solution : Volume of the magnet = $\frac{\text{Mass}}{\text{density}}$

$$= \frac{400\times10^{-3}}{(8\times10^{-3})\times10^6}$$

$$= 50 \times 10^{-6} \; m^3$$

Magnitude of the magnetic moment $P_m = 2Am^2$

∴ So the intensity of magnetisation,

=

$$I = \frac{\text{Magnetic moment}}{\text{volume}} = \frac{2}{50\times10^{-6}}$$

$$M = 0.4 \times 10^5 \; Am^{-3}$$

4. Region $0 \leq z \leq 2m$ is occupied by an infinite stabs of permeable material (μ_r = 3.5). If $\vec{B} = 10y\hat{i} - 5x\hat{j} mwb/m^2$ within the slap determine (a) $\vec{J}$ (b) $\vec{J}_b$ (c) $\vec{M}$ (d) $\vec{K}_b$ on z = 0

Solution : By definition

$$\text{(a)} \quad \vec{J} = \vec{\nabla}\times\vec{H} = \vec{\nabla}\times\frac{\vec{B}}{\mu_0\mu_r} = \frac{1}{4\pi\times10^{-7}(3.5)}\left(\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y}\right)\hat{k}$$

$$= -3.410\hat{k} \; KA/m^2$$

(b) Bound current density,

$$\vec{J}_b = \chi_m\vec{J} = (3.5-1)\,(-3.410) \times 10^3$$

$$= -8.525\hat{k} \; KA/m^2$$

(c) $\vec{M} = \chi_m H = \dfrac{2.5(10y\hat{i} - 5x\hat{j}) \times 10^{-3}}{4\pi\ 10^{-7} \times 3.5}$

$= (5.676\ y\hat{i} - 2.840\hat{j})$ KA/m

(d) $\vec{K}_b = \vec{M} \times \hat{n}$, since z = 0, is the lowerside of the slab occupying $0 \leq z \leq 2$, $\hat{n} = -\hat{k}$ Hence,

$\vec{K}_b = (5.676y\hat{i} - 2.840 \times g) \times (-\hat{k})$

$= (2.840x\hat{i} + 5.676y\hat{j})$ KA/m

5. The volume of the core of a transformer is 1000cc. It is fed with ac if 50 HZ. The loss of energy due to hysteresis. Calculate the area of the B–H loop. CU-13

Solution :

The energy loss per second in the transformer core = $\dfrac{36\text{J}}{3600\text{s}}$

The energy loss per cycle $= \dfrac{36\text{J}}{3600\text{s}} \times \dfrac{1\text{s}}{50\text{cycles}}$

So the energy loss per m^3 = $\dfrac{\dfrac{36}{3600 \times 50^{\text{J}}}}{1000 \times 10^{-6}\text{cycle}}\text{m}^3$

Now, energy loss (in ergs) percycle per cc

$$\frac{1}{4\pi} \times (\text{loop area}) = \frac{36 \times 10^3 \times 10^7}{3600 \times 50 \times 10^6}$$

So, loop area = 25.13 cm^2.

Unit 5 ❑ Electromagnetic Induction

Structure

5.1 Objectives

In this unit, you will study the nature of electromagnetic induction through different related phenomenon as detailed below :

- Concept of magnetic flux.
- Faraday and Neuman's law, and Lenz's law, its application, its importance and specific characteristics.
- Motional electromotive force and Faraday's electromotive force, its quantitative significance.
- What is self inductance and mutual inductance, the ways to measure it and detail aspects to understand.

- Idea about Neuman's expression.
- Arrangement of inducetance in series/parallel, different values inductance to achieve for specific uses.
- Coupling phenomenon in inductance and its application.
- The nature of electromagnetic energy, what is its application in different areas of study.

5.2 Introduction

In 1820, Oersted had shown that an electric current generates a magnetic field. But can a magnetic field generate an electric current? This was almost simultanceously and independently in 1831 by Joseph Henry and Michael Faraday. Faraday showed experimentally that whenever the magnetic flux linked with a closed circuit changes with time an electric current is induced in the circuit. The reason behind the generation of current flow in a closed circuit without any current generating source in it, flow of electric current by changing magnetic flux with time across the loop, is called electromagnetic induction. Faraday's law along with Lenz's law, which follows from conservation of energy, comprise the governing laws of inductive current and its direction. Scientist Neuman, had further elaborated the spectrum of any form of electromagnetic flux flow. This is known as Faraday. Neuman law of electromagnetic induction.

Farady explained electromagnetic induction using the concept of lines of force later on Maxwell used Faraday's ideas and build the foundation of his quantitative electromagnetic theory. Faraday's law has played an important role in the technological transformation as we find today.

5.3 Faraday's Laws of Electromagnetic Induction

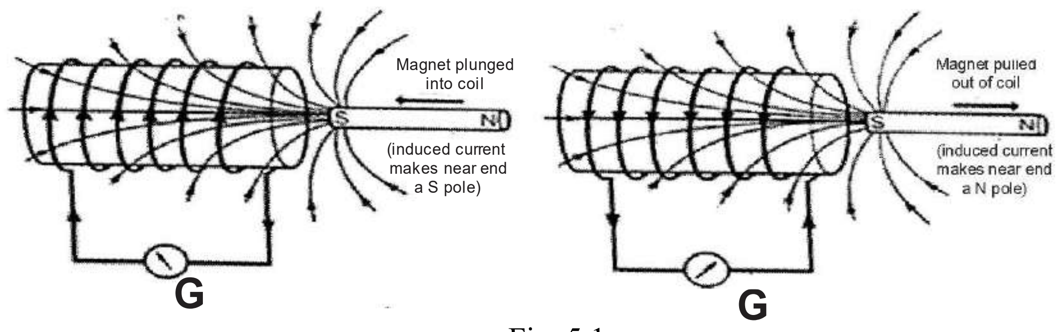


Fig. 5.1

In Fig. 5.1 experiment of Faraday when the bar magnet is moved with respect to the coil following observations have been seen.

i) The galvanometer shows a deflection when ever there is a relative motion between the coil and the magnet. Deflection indicates that an induced current has been set-up in the coil.

ii) Faster movement of the magnet induces more deflection and less when movement of the magnet is slowed.

iii) Reverse deflection in the galvanometer when the same pole is moved in opposite direction or opposite pole of magnet is move in the same direction. The observations led to the inculcation of the following two laws of electromagnetic induction.

a) Induced emf in a circuit is proportional to the rate of change of magnetic flux linked with the circuit.

b) The direction of induced emf is such that it tries to oppose the cause of generation i.e. the variation of magnetic flux inducing it.

The second law is known as Lenz's law, which specifies the direction of current Lenz's law follows from the principle of conservation of energy.

If ϕ is the flux linked with a circuit at any instant t, then $d\phi$ is the time rate of change of flux. The combination of the two laws of electromagnetic induction reveals

$$\varepsilon = -\frac{d\phi}{dt} \qquad \text{........(5.3.1)}$$

where ε is the induced emf. The negative sign indicates that the emf ε opposes the changes of flux. If R is the resistance of the circuit, we get the induced current as

$$i = \frac{e}{R} = -\frac{1}{R}\frac{d\phi}{df} \qquad \text{........(5.3.2)}$$

If the electric field in space is denoted by $\vec{E}$ then emf acound a closed path or curve c is

$$\varepsilon = \oint_e \vec{E}.\vec{dl} \qquad \text{.....(5.3.2)}$$

If S is an open surface bounded by the curve placed in magnetic field $\vec{B}$, then the magnetic flux through the surface

$$\phi = \int_s \vec{B}.\overrightarrow{ds} \qquad ....(5.3.4)$$

Now, using equations (5.3.1), (5.3.2) and (5.3.4) we can write as

$$\oint_c \vec{E}.\overrightarrow{dl} = -\frac{d}{dt}\int_s \vec{B}.\overrightarrow{ds} \qquad ....(5.3.5)$$

which is the intgegral form of Faraday's law when the circuit is fixed, the time derivation can be taken outside the integral, when it becomes partial derivative. Now using stokes theorem, we get,

$$\int_s \left(\vec{\nabla}\times\vec{E}\right).\vec{d}s = -\left(\frac{\partial B}{\partial t}\right).\overrightarrow{ds} \qquad ....(5.3.6)$$

since this must be true for any arbitrary surface s, then we get

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad ....(5.3.7)$$

which is the differential form of Faraday's law.

5.4 Self-inductance

The induced emf ε in a coil is proportional to the rate of change of magnetic flux passing through it due to its own current. This emf is termed as self induced EMF. Magnetic flux produced by the current depends on the geometry of the circuit for non-ferromagnetic material.

The induced emf is proportional to the rate of change of the current through the coil and its proportionality constant is called self-inductance L. If I is the current flowing in a circuit, then associated magnetic flux can be written as,

$$\phi = LI \qquad .....(5.3.8)$$

$$\frac{d\phi}{dt} = \frac{d\phi}{dI}\frac{dI}{dt} \qquad ....(5.3.9)$$

The induced emf in the circuit is given by

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{LdI}{dt} \qquad ....(5.3.10)$$

where $L = \frac{d\phi}{dI}$

The SI unit of self inductance is henry (H). One henry is the value of self-inductance in a closed circuit or coil in which one volt is produced by a variation of the inducing current of one ampere per second. Otherwise a circuit is said to have a self inductance of 1 henry if 1 weber of flux is linked with the circuit never 1 ampere of current flows through it. In SI unit ϕ, I, t, ε and L are expressed in weber, ampere, volt and henry

$$1 \text{ henry} = \frac{1 \text{ weber}}{1 \text{ ampere}} = \frac{1 \text{ vall, second}}{1 \text{ ampere}}$$

$$= 1 \text{ V.A}^{-1}\text{.S}$$

As ϕ has the dimensions $[ML^3T^{-2}I^{-1}]$ and L are of dimensions $[ML^2T^{-2}I^{-2}]$

5.4.1 Calculation of Self-inductance

1. A solenoid : If I be current flow along aircored long solenoid of length containing N number of turns the axial magnetic field at any inside point.

$$\vec{B} = \frac{\mu_0 N.I}{l} \qquad ....(5.4.1)$$

If A is the area of cross-section of the solenoid the flux linking each turn is

$$\phi_1 = BA = \frac{\mu_0 NIA}{l} \qquad ......(5.4.2)$$

and the total flux linking N turns

$$\phi = N\phi_1 = \frac{\mu_0 N^2 AI}{l} \qquad ....(5.4.3)$$

Now the self-inductance L is defined as the flux linkage per unit current. So the self-inductance of the solenoid is

$$L = \frac{\phi}{I} = \frac{\mu_0 N^2 A}{l} \qquad ......(5.4.4)$$

If the solenoid is wound on a materials of permeability μ, then

$$L = \frac{\mu N^2 A}{l} \qquad ....(5.4.5)$$

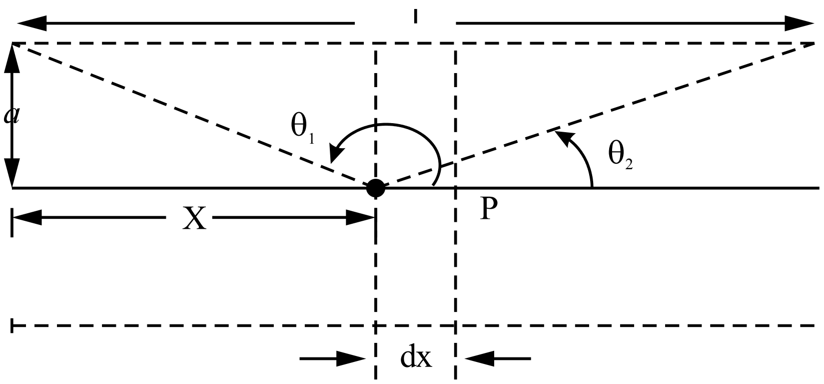


Fig. 5.2

If the solenoid is not very long then the anial magnetic field at any anial point p as show in Fig. 5.2 is given by

$$B = \frac{\mu_0 NI}{2}(\cos\theta_2 - \cos\theta_1)$$

$$= \frac{\mu_0 NI}{2l}\left[\frac{\ell - x}{\sqrt{(\ell - x)^2 + a^2}} + \frac{x}{\sqrt{\left(x^2 + a^2\right)}}\right]$$

In a length dx about P, there are $\frac{N}{L}$dx number of turns and hence the flux linking these turns is

$$d\phi = \left(\frac{N}{l}dx\right)BA$$

So the total magnetic flux through the solenoid is

$$\phi = \int d\phi = \int_0^l \frac{BAN}{l}dx$$

$$= \frac{\mu_0 N^2 AI}{2\ell^2}\int_0^\ell\left[\frac{\ell - x}{\sqrt{(\ell - x)^2 + a^2}} + \frac{x}{\sqrt{(x^2 + a^2)}}\right]$$

$$\text{or,}\quad \phi = \frac{\mu_0 N^2 AI}{2\ell^2}\left[-\sqrt{(\ell - x)^2 + a^2} + \sqrt{x^2 + a^2}\right]_0^l$$

$$= \frac{\mu_0 N^2 A}{\ell^2}\left[\sqrt{(a^2+\ell^2)} - a\right].I$$

So, $$L = \frac{\mu_0 N^2 A}{\ell^2}\left[\sqrt{a^2+\ell^2} - a\right] \qquad ....(5.4.6)$$

$$= \frac{\mu_0 N^2 A}{l}\left[\sqrt{1+(a/\ell)^2} - a/\ell\right]$$

Note that for ℓ>>a, equation (5.4.6) reduces to equation (5.4.5)

2. Long Caonial Cable : Consider a long coaxial cable consisting of two concentric cylinder of inner radius a and outer radius b as shown in Fig. 5.3. The two cylinder carry the same current I in the opposite directions; then they form a coaxial cable.

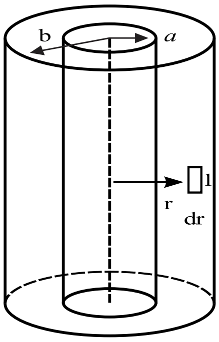


Fig. 5.3

Applying Ampere's circuital law, it can be shown that the magnetic field outside the cable is zero, while at an internal point at a distance r from the axis (a<r<b) the magnetic field is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

If we imagine two coaxial cylinders of radii r and r+dr and of unit length, the flux in the region between the two cylinders is B×(1×dr) = Bdr so the total flux is

$$\phi = \int_{r-a}^{b} B.dr = \frac{\mu_0 I}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0}{2\pi}\ell_n b/a$$

The inductance per unit length is

$$L = \frac{\mu_0}{2\pi} \ln b/a \qquad ....(5.4.7)$$

In the above discussion we have neglected the flux within the materials of the two cylinders. This is justified when b>>a.

If instead of air, the space between the two cylinder is having a medium of magnetic per meability μ, then from equation (5.4.7) will be modified with inductance

$$L = \frac{\mu}{2\pi} \ln b/a \qquad ......(5.4.8)$$

3. Two-wire transmission lines : Two parallel wire transmission line is shown in Fig. 5.4, given the separating distance d, a its radius, and μ is the permeability of the medium in which they reside.

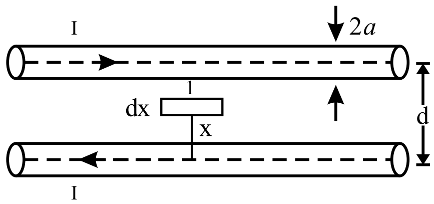


Fig. 5.4

We assume that the radius a of each wire is much less than d, so that the flux inside the material of the wires may be neglected. The two wires carry the same current in the opposite directions. The flux is concentrated between the two wires. Total magnetic field at any point at a distance x from one wire is

$$B = \frac{\mu I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]$$

So the flux through an elemental area of width dx and length unity is

$$d\phi = B.dx \times 1 = \frac{\mu I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx$$

Therefore the total flux through the entire area between the two wires of unit length is

$$\phi = \int d\phi = \frac{\mu_0 I}{2\pi}\int_a^{d-a}\left[\frac{1}{x} + \frac{1}{d-x}\right]dx$$

$$= \frac{\mu I}{2\pi}\left[\ln\frac{d-a}{a} - \ln\frac{a}{d-a}\right]$$

$$= \frac{\mu I}{\pi} \ln \frac{d-a}{a}$$

So, self-inductance per unit length is

$$h = \frac{\phi}{I} = \frac{\mu}{\pi} \ln\left(\frac{d-a}{a}\right) \qquad \text{......(5.4.9)}$$

Assuming d>>a

$$\text{or,} \;\; L \approx \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) H/m$$

4. Toroidal Coil : The magnetic field inside a toroidal coil having mean length L, N being the number of turns of cross-sectional area A, and carrying current I is given by

$$B = \mu n I = \mu \frac{N}{L} I$$

where μ is the permeability of inside medium.

Therefore, the flux through the N turn, neglecting the variation over the cross section, is given by

$$\phi = NBA = \frac{\mu N^2 A}{L} I$$

So the self-inductance is

$$L = \frac{\phi}{I} = \frac{\mu N^2 A}{L} \qquad \text{.......(5.4.10)}$$

5.5 Mutual Inductance

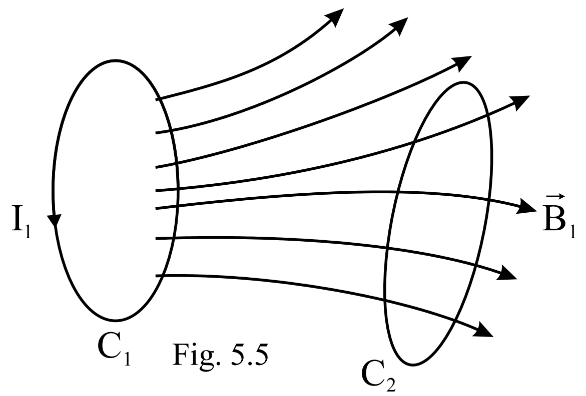


Fig. 5.5

Two coils C_1 and C_2 are two fixed coils placed sufficiently close to each other, as shown in Fig. 5.5. If I_1 is current passed through the coil C_1 then magnetic field B_1 will be produced around the coil C_1. Magnetic flux ϕ_{21} will be passed through the coil C_2 due to B_1. Alternatively, we can say that ϕ_{21} flux linkage due to B_1 the magnetic field of coil C_1.

B_1 will change if I_1 changes, then ϕ_1 and ϕ_{21} will also change. An induced emf will be produced in coil C_2 due to this. This phenomenon is called mutual induction.

Again, there will be change in ϕ_1, due to the change in current I, as a consequence an induced emf will also be induced in C_1. This is called self-induction.

For a number of turns in both the coils, we can write

$$\phi_{21} = \int_{s_2} \vec{B}_1.\overrightarrow{ds_2}$$

where $d\vec{s}_2$ is the elemental area in the coil 2. By Bio t-Savart law we can write

$$\vec{B} = \frac{\mu_0 I_1}{4\pi} \oint \frac{\overrightarrow{d\ell} \times \hat{r}}{r^2}$$

So for coil C_1, $\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{\overrightarrow{d\ell_1} \times \hat{r}}{r^2}$

Taking into consideration that, other features of the coil as intact, $\vec{B}_1$ depends only on I, so we can write,

$$\phi_{21} = \frac{\mu_0 I_1}{4\pi} \int_{s_2} \oint_{l_1} \frac{\overrightarrow{d\ell_1} \times \hat{r}.\overrightarrow{ds_2}}{r^2}$$

$$\mu\phi_{21} = M_{21}\ I_1 \qquad \text{........(5.5.1)}$$

nowhere $M_{2\ell} = \frac{\mu_0}{4\pi} \int_{s_2} \oint_{\ell_1} \frac{\overrightarrow{d\ell_1} \times \hat{r}.\overrightarrow{ds_2}}{r^2}$ ......(5.5.2)

M_{21} is proportionality constant between ϕ_{21} and I_1. Induced emf in the coil C_2, according to Faraday's law, will be

$$\varepsilon_2 = -\frac{d\phi_{2\ell}}{dt} = -M_{2\ell}\frac{dI\ell}{dt} \qquad \text{........(5.5.3)}$$

It is observed from the relation in equation (5.5.3) that induced emf in coil C_2 is related to the current flow changes in coil C_1 M_{21} is defined as co-efficient of mutual inductance. It will be kept in mind that this will remain unchanged if the configuration of the two coil remain fixed.

5.6 Neumann's Formula

Determination of mutual inductance is very complex, depending on the set up of two coils Neumann has formulated a relation to simply the calculation. We know that flux linkage is given by

$$\phi_{21} = \int_{s_2} \vec{B}_1.\overrightarrow{ds}_2 \text{ and } \vec{B}_1 = \vec{\nabla} \times \vec{A}_1$$

where $\vec{A}_1$ is the magnetic vector potential corresponding to $\vec{B}_1$. Also, we know that vector potential is given by

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint_c \frac{\overrightarrow{dl}}{r}$$

or, $$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{c_1} \frac{\overrightarrow{dl}}{r_1}$$

Hence $$\phi_{21} = \int_{s_2} \left(\vec{\nabla} \times \vec{A}\right).\overrightarrow{ds}_2 = \oint_{c_2} \vec{A}_1.\overrightarrow{dl}_2$$

or, $$\phi_{21} = \frac{\mu_0 I_1}{4\pi} \oint_{c_2} \left[\oint_{c_1} \frac{\overrightarrow{dl}_1}{r}\right].\overrightarrow{dl}_2$$

But we know, $\phi_{21} = M_{21} \; I_1$

$$\therefore \quad M_{21} = \frac{\mu_0}{4\pi} \oint_{c_2} \oint_{c_1} \frac{\overrightarrow{dl}_1.\overrightarrow{dl}_2}{r} \qquad ....(5.6.1)$$

Since the order of integration may be interchanged we can write

$$M_{21} = M_{12} = \frac{\mu_0}{4\pi} \oint \oint \frac{\overrightarrow{dl}_1.\overrightarrow{dl}_2}{r} \qquad ....(5.6.2)$$

This is known as Neumann's formula for the mutual inductance of two arbitrary coils or loops. The double integral (5.6.2) is not easy to work with except for circuits. With simple geometry but it does illuminate two important points :

i) $M_{12}=M_{21}=M$. This signifies that in any case the flux ϕ_1 through loop C_1 when a current I flows around C_2 is exactly equal to the flux ϕ_2 through loop C_2 when the same current I flows around C_1. This is called as reciprocity theorem.

ii) M_{12} or M_{21} is depends on the structure of the coil, configuration and relative position of the two coils.

5.7 Calculation of Mutual Inductance

1. Two solenoids :

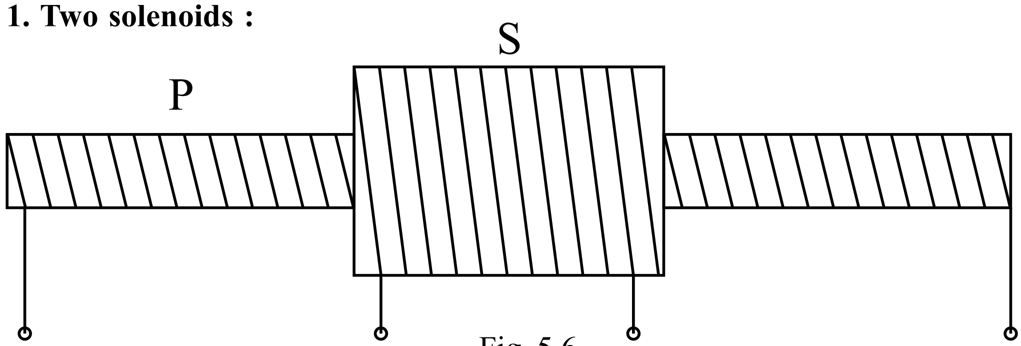


Fig. 5.6

Two coaxial solenoids are shown in the Fig. 5.6 where P is a long primary solenoid and S is short secondary solenoid. There is alomost no magnetic field outside the long solenoid. If a current I flows through the primary, the magnetic induction produced at the centre would be

$$B = \mu_0 nI = \mu_0 \frac{N_1}{L} I$$

where N_1, and L were the total number of turns and length of the primary solenoid, respectively. If A be cross-sectional area of P, then flux linked with the secondary coil of total number of turns N_2 would be

$$\phi = B.A.N_2 = \frac{\mu_0 N_1 N_2 A}{L} I$$

So the mutual inductance will be $M = \frac{\phi}{I} = \frac{\mu_0 \times N_2 N_{IA}}{L}$ ......(5.7.1)

2. Two parallel circular coaxial coil

Let C_1 and C_2 are two parallel circular coanial coils with the centres O_1 and O_2 and radii a and b (Fig. 5.7). x is the axial separation O_1 and O_2. The flux through C_2 can be assumed to be uniform taking into account C_2 is small compared to C_1. If I is the current in C_1, magnetic induction at O_2 is given by

$$B = \frac{\mu_0 2\pi N_1 I a^2}{4\pi(a^2 + x^2)^{3/2}}$$

where N_1 is the number of turns in coil C_1. Total flux linked with the coil C_2 is

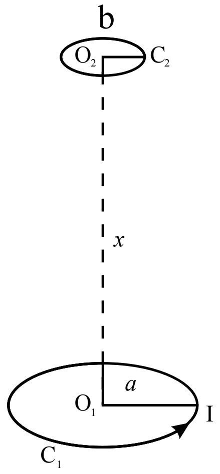


Fig. 5.7

$$\phi_2 = B.\ \pi b^2 N_2$$

$$\text{or,} \quad \phi_2 = \frac{\mu_0 \pi N_1 N_2}{2(a^2 + x^2)^{3/2}} a^2 b^2 I \qquad ....(5.7.2)$$

So the mutual inductance between the coil is

$$M = \frac{\phi_2}{I} = \frac{\mu_0 I N_1 N_2 a^2 b^2}{2(a^2 + x^2)^{3/2}} \qquad ....(5.7.3)$$

If the coils are coplanar then x = 0 and $M = \frac{\mu_0 \pi N_1 N_2 b^2}{2a}$

Value of M for large circular loop C_2 can be determined by using Neumann's formula.

5.8 Inductance in series and parallel combinations of Inductances

1. Series connection :

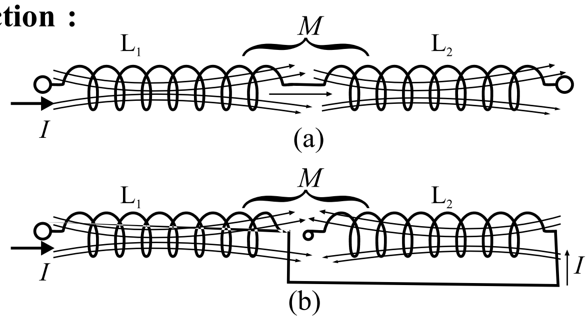


Fig. 5.8

Fig. 5.8 (a) shows two coils of self-inductances connected in series. The induced emf in coil 1 due to self-inductance when current I flows through it,

$$\varepsilon_{11} = L_1 \frac{dI}{dt}$$

while the emf induced in coil 2 due to the current I in coil 1 is $\varepsilon_{21} = -M\frac{dI}{dt}$

where M is the mutual inductance of the two coils. The emf induced in the coil due to self inductance is $\varepsilon_{22} = -L_2\frac{dI}{dt}$ and the emf in coil 1 due to the current in coil 2 is $\varepsilon_{12} = -M\frac{dI}{dt}$

Total emf due to the flux aiding me and another is

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{12} + \varepsilon_{21} = -(L_1 + L_2 + 2M)\frac{dI}{dt} \qquad .....(5.8.1)$$

Again, $$\varepsilon = -L_{eq}\frac{dI}{dt} \qquad ....(5.8.2)$$

Comparing equations (5.8.1) and (5.8.2) the equivalent self-inductance, L_{eq} is

$$L_{eq} = L_1 + L_2 + 2M \qquad ....(5.8.3)$$

In Fig. 5.8(b) mutual flux opposes the self-flux of the two coil in series, then we get,

$$\varepsilon = \varepsilon_{11} + \varepsilon_{22} - \varepsilon_{12} - \varepsilon_{21}$$

So the equivalent self-inductance is

$$L_{eq} = L_1 + L_2 - 2M \qquad ......(5.8.4)$$

2. Parallel connection

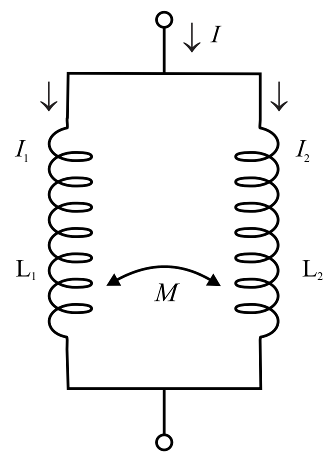


Fig. 5.9

Fig. 5.9 shows two coils of self-inductances L_1 and L_2 connected in parallel. Total I gets divided into branches as I_1 and I_2. Assuming that the mutual flux aids the self-flux, the total emf induced in coil 1 is

$$\varepsilon_1 = -L_1 \frac{dI}{dt} - M\frac{dI}{dt}$$

Similarly

$$\varepsilon_2 = -L_2 \frac{dI_2}{dt} - M\frac{dI_1}{dt}$$

Since the two coils have the same emf i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon$ for being parallel, we have

$$L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} = -\varepsilon \qquad \text{......(5.8.5)}$$

and $$\frac{MdI_2}{dt} + \frac{L_2 dI_2}{dt} = -\varepsilon \qquad \text{......(5.8.6)}$$

Solving this two equations, we get

$$\frac{dI_1}{dt} = \frac{-\varepsilon(L_2 - M)}{L_1L_2 - M^2} \text{ and } \frac{dI_2}{dt} = \frac{-\varepsilon(L_1 - M)}{L_1L_2 - M^2}$$

Therefore, $\dfrac{dI}{dt} = \dfrac{dI_1}{dt} + \dfrac{dI_2}{dt} = \dfrac{-\varepsilon(L_1 + L_2 - 2M)}{L_1L_2 - M^2}$

If L_{eq} be the equivalent self-inductance, then $\varepsilon = -L_{eq}\dfrac{dI}{dt}$

$$\therefore \; L_{eq} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M} \qquad \text{......(5.8.7)}$$

If there is no magnetic coupling between the coils then M = 0 and we have

$$L_{eq} = \frac{L_1L_2}{L_1 + L_2} \text{ or, } \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \qquad \text{.......(5.5.8)}$$

3. Coefficient of coupling : In order to find mutual inductance, there is necessity for two disconnected coil, so that current flow in one coil can induce emf on other coil. Mutual flux between them can be less than or at best equal to the self-fluxes of the two loops. It implies that $\phi_{12} \leq \phi_{22}$ and $\phi_{21} \leq \phi_{11}$. So we can write $\phi_{12} = K_1\phi_{22}$ and $\phi_{21} = K_2\phi_{11}$, where K_1 and K_2 are two numbers less than or equal to one. So we can write,

$$MI_2 = \phi_{12} = K_1\phi_{22} = K_1L_2I_2$$

since $\phi_{22} = L_2I_2$

Hence, $M = K_1L_2$ ....(5.8.9)

$$M.I_1 = \phi_{21} = K_2\phi_{11} = K_2L_1I_1$$

or, $M = K_2L_1$ ....(5.8.10)

From equations (5.8.9) and (5.8.10) we get,

$$M^2 = K_1K_2L_1L_2$$

$$\text{or,} \quad M = K\sqrt{L_1L_2} \qquad ....(5.8.11)$$

where $K = K_1K_2$, and $0 < K \leq 1$. This geometrical constant is known as coefficient of coupling of the loops. This coupling coefficient depends on verying geometry, which can be designed according to one's criteria.

5.9 Magnetic Energy

1. Energy in an inductor : When a electric current flows in an inductor it will store energy in the form magnetic field. For a pure conductor power which must be supplied at any instant of time to initial current through the inductor is

$$P = iv = Li\frac{di}{dt}$$

Hence the energy in put to have a final current i is given by—

$$\text{Energy stored (E)} = \int_0^t Pdt = \int_0^1 Lidi$$

$$\therefore E = \frac{1}{2}LI^2 \qquad ....(5.9.1)$$

Self-inductance of a circuit can be defined as the two times the magnetic energy stored in a circuit when a unit current is established in it. So the self-inductance is thus a measure of the magnetic energy stored in the circuit for a given current.

2. Energy stored in a magnetic field : Energy is required to establish a magnetic field, which is stored as a magnetic field energy. Let us take a number of current carrying loops in a finite region of a medium. Magnetic flux associated with the ith circuit is given by

$$\phi_i = \int_{s_i} \vec{B}.\hat{n}ds = \oint_{c_i} \vec{A}.\vec{dl}_i \qquad ......(5.9.2)$$

where A is the magnetic vector potential associated to $\vec{B}$ by $\vec{B} = \vec{\nabla} \times \vec{A}$. The magnetic energy of the system is

$$U = \frac{1}{2}\sum_{i=1}^{n} I_i\phi_i$$

$$= \frac{1}{2}\sum_{il}\oint I_i\vec{A}_i.d\vec{l}_i \qquad ....(5.9.3)$$

Assuming each circuit is a closed path in the medium which is conducting. $I_i\vec{dl}_i$ should be replaced by $\vec{J}dv$ and $\sum_i \oint_{c_i}$ by $\int_v$ so

$$U = \frac{1}{2}\int_v \vec{J}.\vec{A}dV \qquad ....(5.9.4)$$

where $\vec{J}$ is the volume current density. Now using the relation $\vec{\nabla} \times \vec{M} = \vec{J}$, we can write equation (5.9.4) as

$$U = \frac{1}{2}\int_v (\vec{\nabla}\times\vec{H}).\vec{A}dV \qquad ....(5.9.5)$$

From, vector identity

$$\vec{\nabla}.(\vec{A}\times\vec{H}) = \vec{H}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{H})$$

Now, using the vector identity and divergence theorem, we get

$$U = \frac{1}{2}\int_V \vec{H}.(\vec{\nabla}\times\vec{A})dV - \frac{1}{2}\int_S \vec{A}\times\vec{H}.\hat{n}ds \qquad ....(5.9.6)$$

where S is the surface bounding the volume V.

Note that integration is to be carried out over the entire volume occupied by the current distribution. For convenience, the surfaces can be moved to infinity. This will not affect the integration (5.9.4) because $\vec{J} = 0$ outside the region occupied by the current distribution. Now at large distances $\vec{A}$ and $\vec{H}$ will fall at least as rapidly as those of dipoles, hence $A \sim \frac{1}{r^2}$, $H \sim \frac{1}{r^3}$. So the integrand $\vec{A} \times \vec{H}$ falls off at least as $^1/r^3$ or faster. As the surface element $d\vec{s}$ goes as r^2, the surface integral vanishes as 1/r or faster as r goes to infinity.

Therefore, equation (5.9.6) becomes $U = \frac{1}{2}\int \vec{H}.\vec{B}dv$ ......(5.9.7)

So we conclude from equation (5.9.7) that magnetic energy stored in a magnetic field with energy density as $\frac{1}{2}(\vec{H}.\vec{B})$. So we can write, energy density as

$$U = \frac{1}{2}\vec{H}.\vec{B} = \frac{B^2}{2\mu} = \frac{1}{2}\mu H^2 \qquad ......(5.9.8)$$

5.10 Summary

We have learned following topics on electromagnetic induction.

1. Idea about magnetic flux : $\phi = BS = \int \vec{B}.\vec{ds}$ Faraday and Neumann's laws $\varepsilon = \frac{d\phi}{dt}$ and Lenz's law.

2. Differential form of Faraday's law $\vec{\nabla}\times\vec{B} = -\frac{\partial B}{\partial t}$ and integral form $\varepsilon = -\frac{d\phi}{dt} = \oint \vec{E}.\vec{dl}$

3. Self-inducetance and mutual inductance. Inducetance L = $\frac{\phi}{I}$, and mutual inductance $M = \frac{\phi_2}{I_1} = \frac{\phi_1}{I_2}$.

4. Neumann's expression far mutual inductance :

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{c_2} \oint_{c_1} \frac{\vec{dl}_1 . \vec{dl}_2}{r} = M_{12}$$

We have studied self and mutual inductance of current loop with different geometrical shapes.

5. Magnetic energy : $U_{mag} = \frac{1}{2} LI^2 = \frac{1}{2} \int \vec{A}.\vec{J} dv$

$$= \frac{1}{2\mu_0} \int B^2 dv$$

6. Energy density in magnetic field

$$u = \frac{\vec{H}.\vec{B}}{2} = \frac{B^2}{2\mu_0} = \frac{\mu_0 H^2}{2}$$

5.11 Review Questions and Answers

1. Obtain the integral form of Faraday's law and then show that

$$\vec{\nabla} \times \vec{E} = -\frac{\overrightarrow{\partial B}}{\partial t}$$

Answer : See article 5.3.

2. Starting from energy consideration prove that $M^2 \leq L_1 L_2$.

Answer : For two fixed closed circuit with positive coupling, with currents C_1 and C_2 in the respective circuits

EME equations at any instant.

$$\varepsilon_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \qquad \text{........(5.9.1)}$$

$$\varepsilon_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \qquad \text{.......(5.9.2)}$$

From energy conservation consideration, the rate of energy supplied from the source must be equal to the rate of Joule heat dissipation plus the rate of energy stored in the magnetic field. so,

$$\varepsilon_1 i_1 + \varepsilon_2 i_2 = i_1^2 R_1 + i_2^2 R_2 + \frac{dU}{dt} mag$$

$$\frac{dU}{dt} mag = \varepsilon_1 i_1^1 + \varepsilon_2 i_2^1 - \left(i_1^2 R_1 + i_2^2 R_2\right)$$
$$= \varepsilon_1 i_1 + e_2 i_2 - (ik_1^2 + i_2^2 k_2)$$

From (5.9.1) and (5.9.2), We get

$$\frac{dU_{mag}}{dt} = L_1 i_1 \frac{di_1}{dt} + L_2 i_2 \frac{di_2}{dt} + M_1 i_1 \frac{di_2}{dt} + M i_2 \frac{di_1}{dt}$$

$$= \frac{1}{2} L_1 \frac{di_1^2}{dt} + \frac{1}{2} L_2 \frac{di_2^2}{dt} + M \frac{d}{dt}(i_1 i_2)$$

Taking $i_1 = i_2 = 0$ at t = 0, and integrating the equation from t = 0, to t = t We find

$$U_{mag} = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L \ \ i_2 + M i_1 i_2 \qquad \text{........(5.9.3)}$$

$$\text{or,} \quad \frac{2U_{mag}}{L_1} = \left(i_1 + \frac{M}{L_1} i_2\right)^2 + \left(\frac{L_2}{L_1} - \frac{M^2}{L_1}\right) i_2^2 \qquad \text{........(5.9.4)}$$

which is valid for all i_1 and i_2 and let $i_1 = -\frac{M}{L_1} i_2$. Then since U_{mag} is positive or zero for all values of i_1 and i_2 we must have $M^2 \leq L_1 L_2$.

3. Show that the equivalent inductance of the two coils of self-inductances L_1 and L_2, connected in parallel is

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm M}$$

Answer : See articale (5.8) for answer.

4. Show that the self-inductance of a long solenoid length l radius a and with n turns per unit length is approximately given by

$$L = \mu_0 n^2 . \pi a^2 \left[\sqrt{l^2 + a^2} - a \right]$$

Answer : See Article 5.4.

5. State and prove the reciprocity theorem in mutual inductance. Derive Neumann's formula for the mutual inductance between two arbitrary loops.

Answer: See article 5.6.

6. Two long parallel wires carrying the same current I in the opposite direction and separated by a distance d in the air. The length of the wire are much larger than d. Find the self-inductance per unit length.

Answer : See article 5.4.

7. Two coils with self-inductance L_1 and L_2 respectively, have mutual inductance M. Find an expression for their coefficient of coupling.

Answer : See article 5.8.

8. Obtain a formula for the mutual induction between two loops carrying current.

Answer : See article 5.7.

5.12 Problems and Solutions

1. A wire of length 1m moves at right angle to its length at a speed of 100 m/s in a uniform magnetic field 1 wb/m² which is also acting at right angle to the length of the wire. Calculate the emf induced in the wire when the direction of motion— (i) right angles to the field, (ii) inclined at 30° to the field.

Solution :

Induced electric field due to motion in magnetic field is equal $\vec{E} = (\vec{\nu} \times \vec{B})$

Induced emf $\varepsilon = \oint \vec{E}.\vec{dl}$

$$= \oint (\vec{\nu} \times \vec{B}).dl$$

For a length 'L' of the rod, induced emf will be

$\varepsilon = vBL\sin\theta$ (θ is the angle between $\vec{v}$ and $\vec{B}$)

(i) Here $\theta = 90°$ $\therefore$ $\varepsilon = vBL$

which $v = 100\ ms^{-1}$, $B = 1wbm^{-2}$, $L = 1m$

$\therefore$ $\varepsilon = 100volt$

(ii) When $\theta = 30°$

$$\varepsilon = \frac{vBL}{2} = 50V$$

2. A conducting metallic disc is rotating about an axis passing through its centre, perpendicular to its own plane. An external magnetic field is applied in a direction perpendicular to the plane of the disc. What will induced emf ? What will be the current flow it a metallic wire is connected between periphery and the axis?

Solution : Let P be point where the wire is connected at the periphery. Let the disc rototates, it position at t is covers a distance dr in time. $t + \delta t$. i.e. PQ = dr. So the area POQ = $\frac{1}{2}$rddr. Intercepted magnetic flux $d\phi = \frac{1}{2}$ Brdr.

Therefore the induced emf between O and P

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{1}{2}Br\frac{dr}{dt} = \frac{1}{2}Brv$$

or, $\varepsilon = \frac{1}{2}Br^2w$

The direction of the induced emf cannot be determined in this specifi case.

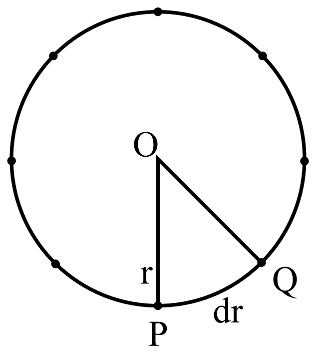


Fig. 3

Current flow $I = \frac{\varepsilon}{R} = \frac{Br^2\omega}{2R}$

3. Suppose a square loop of side a is placed in the plane of a long straight wire carrying current I. The nearest side of the loop is at a distance r from the wire. Find the magnetic flex through the loop. If someone pulls the loop directly away from the wire at a constant speed, what should be the emf generated in the loop? What is the value of emf generated when the loop is pulled parallel to the wire?

Solution :

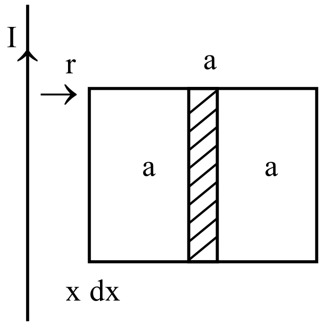


Fig. 4

The magnetic induction at a distance x from the wire is $B=\frac{\mu_0}{4\pi}.\frac{2I}{x}$

This magnetic field is in a direction normal to plane of the loop. So the flux through an elemental area adx within the loop as shown in Fig. 4.

$$d\phi = Badx = \frac{\mu_0 Iadx}{2\pi x}$$

Total flux across the loop is

$$\phi = \frac{\mu_0 Ia}{2\pi}\int_r^{r+a}\frac{dx}{x} = \mu_0\frac{Ia}{2\pi}\frac{r+a}{a}$$

The induced emf in the loop is

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{d\phi}{dr}.\frac{dr}{dt}$$

$$= -\frac{vd\phi}{dr} = -\frac{\mu_0 Iav}{2\pi}\left[\frac{1}{r+a} - {}^{1}\!/_{r}\right]$$

$$= \frac{\mu_0 Iva^2}{2\pi r(r+a)}$$

4. A square loop of wire of side 'a' lies on a table near a very long straight wire which carries a current I. Find the flux through the loop if it moves away from the wire at a speed 'v'. What is the emf generated ? What is the emf generated in case the loop is pulled to the right with the same speed ?

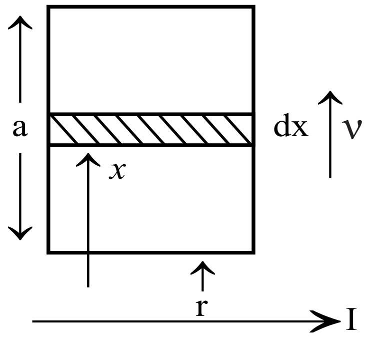


Fig. 6

Solution: The magnetic induction at a distance x from the long wire which carries current I is

$$B = \frac{\mu_0 I}{2\pi x}$$

which is directed normally on the loop the flux through an elementary area adx within the loop is given by

$$d\phi = Badx = \frac{\mu_0 Ia}{2\pi}\frac{dx}{x}$$

or, $$\phi = \frac{a\mu_0 I}{2\pi}\int_r^{a+r}\frac{dx}{x} = \frac{\mu_0 Ia}{2\pi}l_n\left(\frac{a+r}{r}\right)$$

or, $$\phi = \frac{\mu_0 Ia}{2\pi}\ln\left(1+\frac{a}{r}\right)$$

∴ Induced emf

$$\varepsilon = -\frac{d\phi}{dt} = -\frac{d\phi}{dr}.\frac{dr}{dt}$$

$$= -v\frac{d\phi}{dt} = -\frac{\mu_0 Iv}{2\pi}\frac{1}{1+\frac{a}{r}}\left(-\frac{a}{r^2}\right)$$

$$\therefore \varepsilon = \frac{\mu_0 va^2}{2\pi r(r+a)}$$

The current is in anti clockwise direction, when the flux is pulled right, there is no change in flux, so ε = 0.

Unit 6 ❑ Maxwells Equations And Electromagnetic Wave Propagation

Structure

6.1 Objectives

After the completion of the Unit, learners will be able to under stand :

1 Nature of electromagnetic waves and its source

2 Unified classical theory of electromagnetism forwarded by Maxwell and prediction of oscillating nature of electromangnetic waves

3 Electromagnetic spectrum

4 Boundary condition,behavior of electromagnetic wave throgh free space, throgh different media like dielectric, metal etc.

Its behavirol changes while interfacing different media.

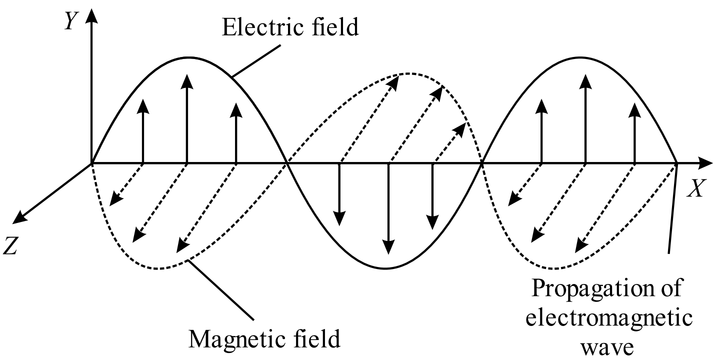
5 Different optical phenomen, like total internal reflection, Brewester's law and evanascent waves etc.

6 Reflection and Refraction of Electromagnetic of waves.

7 Gauge Theories

6.2 Introduction

Electromagnetic waves EM are synchrorized oscilliations of electric and magnetic fields. These waves are created due periodic change of electric and magnetic fields. The creations of electromagnetic waves are formed when a cherged particle is accelerated as a part of oscillatory motion, the charged particle creates ripples on oscillations in its electric field and also produces a magnetic field. The electric and magnetic of the wave are perpendicular to each other and also, perpendicular to the



Electromagnetic Wave

Fig. 6.1

direction of the EM waves.

EM waves carry energy, momentum and angular momentum away from their source particle and can impart those quantites to matter with which they interact them. EM waves travel with a constant velocity of C=3x10^8 m/s in vacuum. They are diflected neither by the electric field nor the magnetic field, they exhibits, interference diffraction. An EM wave can travel through anything be it vacuum air and solid material. It requaires no medium to propagale from one space to another space. This tranverse wave are measured by their amplitude, and wave length. The highest point of a wave is known as 'Crest', whereas the longest point is known as 'Trough'. The EM wave can be split into a range of frequencies, which is known as electromagnetic spectrum-examples are radiowaves, microwaves infrared, x-rays and gamma rays etc.

In quantum mechanics, EM radiowaves, are termed as photons, uncharged elementary particles with zero rest mass which are the quanta of the electromagnetic field responsible for all electromagnetic interactions Quantum effects generates additinal source of radiation such as the transition of electrons to lower energy in an atom and black body radiation photons have energy of 'hν' where h is Planek's constant, the higher the frequency the higher the energy.

6.3 Displacement Current and Maxwell's Equations Displacement Current

From Ampere's law we can calculate the magnetic field due to steady current flow. We know that the integral $\oint \vec{H}.d\vec{l}$ around any closed path or loop is equal to *i*' where *i* is current passing an area bounded by the closed curve C (Fig. 6.2)

If Ampere's law is true all the time, then the *i* determined should be independent of the surface chosen.

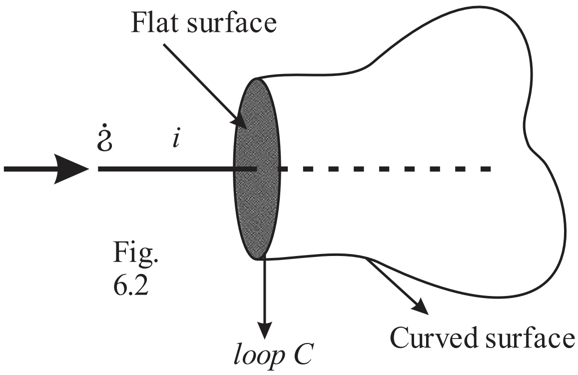
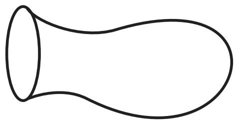


Fig. 6.2

Let us consider the case of charging of a capacitor by (Fig.6.3) charging current $i\ (t) = \frac{V}{R}\ e^{-t/Rc}$, which leads to a magnetic field $\vec{H}$ as observed, with the Ampere's law $\oint \vec{B}.d\vec{l} = i$

If we look at $i = i$ (t) 

If look at $i = O$ 

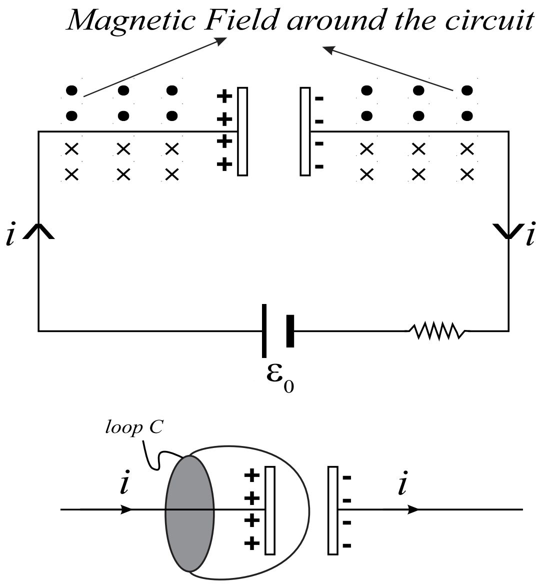


Fig. 6.3

This is because there is no charge flowing between the capacitor plates. It points out that Ampere's law is either wrong or incomplete. Also from Ampere's law in differential form

$\vec{\nabla} \times \vec{H} = \vec{J}$ ....(6.3.1)

Where $\vec{J}$ is the current density. Taking the divergence of the above equation.

$$\vec{\nabla}.\vec{\nabla} \times \vec{H} = \vec{\nabla}.\vec{J} = O$$

This reflects that $\vec{\nabla}.\vec{J} = O$ which violates the continuty equation. As the electric charge is piling up on the plate of the capacitor contained whithin the volume enclosed by the surface S_1 and S_2, the continuity equation is

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = O \quad \text{...(6.3.2)}$$

Where ρ is the charge density on the capacitor which varies with time. So, some quantity must be added to equation (6.3.1) on the right hand side, which must be consistent with the equation (6.3.2). In order to find this, quantity, which must be consistent, an electric displacement vector $\vec{D}$ related to the charge density ρ by

$$\vec{\nabla}.\vec{D} = \rho \quad \text{...(6.3.3)}$$

From equation (6.3.2) and (6.3.1)

We find $\vec{\nabla}.\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{D}) = \vec{\nabla}.\left[J + \frac{\partial D}{\partial t}\right] = O$ ....(6.3.4)

Now if we add $\frac{\partial \vec{D}}{\partial t}$ to the right hand side of equation (6.3.1) then its divergence will satisfy equation (6.3.2) with the inclusion of $\frac{\partial \vec{D}}{\partial t}$, in Ampere's law of differential form, we have, $\vec{\nabla}\times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ ....(6.3.5)

The quantity $\frac{\partial \vec{D}}{\partial t}$ was first introduced by Maxwell and is called displacement current density. For a very slowly varying field $\frac{\partial \vec{D}}{\partial t}$ is negligible. We can use equation (6.3.1) of unomodified Ampere's law of steady field. We learnt earlier that the electric field can be generated by charges and changing magnetic flux. So we see from Ampere Maxwell that a magnetic field can be generated by moving charges (current) and changing electricflux. Thats is a change in electric flux throgh a surface bounded by C can lead to an induced magnetic field along the loop ie: induce magnetic field is along the same direction at caused by the changing electric flux, without the term $J_D = \frac{\partial \vec{D}}{\partial t}$ electromagnetic wave propagation would be impossible. Based on the displacement current density, we define the displacement current, as $I_d = \int \bar{J}_d . \vec{ds} = \int \frac{\partial \vec{D}}{\partial t} . \vec{ds}$

We must bear in mind that displacement current is a result of time varying electric field.

6.4 Maxwell's Equation

Here we sumarize the laws associated with electromagnetic field :

$$\vec{\nabla}.\vec{D} = \rho \quad \text{...(6.4.1)}$$

$$\vec{\nabla}.\vec{B} = O \quad \text{...(6.4.2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{...(6.4.3)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{...(6.4.4)}$$

Equation (6.4.1) and (6.4.2) express Gauss's law for the electric and the magnetic field respectively. Equation (6.4.1) is a mathematical statement of Coulomb's law, while the physical significance of equation (6.4.2) is the absence of free magnetic monopole Equation (6.4.3) states the Faraday-Henry law of electromagnetic induction, while equation (6.4.4) is the Ampere-Maxwell law containing the factor displacement current density. $\frac{\partial D}{\partial t}$

All the equations comprising (6.4.1), (6.4.2), (6.4.3) and (6.4.4) represent Maxwell equations. It is also highlighted that the term $\rho\,(\vec{r}, t)$ and $\vec{J}(\vec{R}, t)$ in all the above equation contain all charges and current respectively, whether free or bound.

There is however, a more convenient form of the set of general equation of Maxwell, suitable for the study of electromagnetic fields inside material subtances that are subject to electrical polarization and magnatizetion, let $(\vec{E}, \vec{B})$ represents electromagnetic field inside the material of subtance having both electric and magnetic properties assuming P as polarization vector and M magnetization vectors respectively Introducing the auxillary fields we have.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}\,,\ \vec{H} = \frac{I}{\mu_0}\vec{B} - \vec{M} \quad \text{.......(6.4.5)}$$

For a linear medium,

$$\vec{P} = \varepsilon\, x_e\, \vec{E}\,,\ \vec{M} = \chi_m \vec{H} \quad \text{.......(6.4.6)}$$

so that $$\vec{D} = \varepsilon\vec{E}\,,\ \vec{H} = \frac{1}{\mu}\,\vec{B} \quad \text{.......(6.4.7)}$$

where, $$\varepsilon = \varepsilon o\,(1+\chi_e) \text{ and } \mu = \mu_o\,(1+\chi_m) \quad \text{.......(6.4.8)}$$

Modified Maxwell equation taking into account $\rho_t(\vec{r}, t)$ and $J_t(\vec{r}, t)$ as the free charge and current densities respectively inside the meterial take the form

$$\vec{\nabla}.\vec{D} = \rho_f \quad \text{.......(6.4.9)}$$

$$\vec{\nabla}.\vec{B} = O \quad \text{.......(6.4.10)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{.......(6.4.11)}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_t + \frac{\partial \vec{D}}{\partial t} \quad \text{...(6.4.12)}$$

with proper set of different boundary condition depending on the different material media at their interface, solution representing their properties, for the above set of equation can be obtained.

The Maxwell equations are also associated with certain conservation laws, such as the conservation of charge and the conservation of energy which is explaned by Poynting's theorem. Conservation of charge can be demonstrated with the help of equation (6.4.1) and (6.4.4). Taking the divergence of equation (6.4.4)

$$\vec{\nabla}.\, \vec{\nabla} \times \vec{H} = \vec{\nabla}.\vec{J} + \vec{\nabla}.\, \frac{\partial \vec{D}}{\partial t}$$

$$O = \vec{\nabla}.\vec{J} + \frac{\partial}{\partial t}\, (\vec{\nabla}.\vec{D})$$

$$O = \vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} \quad \text{..(6.4.13)}$$

Equation (6.4.3) represents the equation of continuity.

Taking the divergence of both side of equation (6.4.3)

$\vec{\nabla}.\, \vec{\nabla} \times \vec{E} = O - \vec{\nabla}.\, \frac{\partial \vec{B}}{\partial t} = = \frac{\partial}{\partial t}(\vec{\nabla}.\vec{B})$, so div $\vec{B}$ = O, which is compatible with equation (6.4.2)

6.5 Poynting's Theorem

It states that in a given volume, the stored energy changes at a rate given by the work done on the charges within the volume, minus the rate of which energy laves the volume.

From equation (6.4.4) and (6.4.5) we can obtain,

$$\vec{H}.\vec{\nabla} \times \vec{E} - \vec{E}.\vec{\nabla} \times \vec{H} = -\vec{H}.\, \frac{\partial \vec{B}}{\partial t} - \vec{E}.\frac{\partial \vec{D}}{\partial t} - \vec{E}.\, \vec{J} \quad \text{..............................(6.5.1)}$$

using the vector identity

$$\vec{\nabla}(\vec{A} \times \vec{B}) = \vec{B}.\ \vec{\nabla} \times \vec{A} - \vec{A}.\vec{\nabla} \times B$$

in the LHS of equation (6.4.13) we have,

$$\vec{\nabla}(\vec{E} \times \vec{H}) = -\vec{H}\frac{\partial\vec{B}}{\partial t} - \vec{E}.\frac{\partial\vec{D}}{\partial t} - \vec{E}.\vec{J} \quad \text{...(6.5.2)}$$

For a linear and non-dispersive media, we can write, $\vec{D} = \varepsilon.\vec{E}$ and $\vec{B} = \mu\vec{H}$ where ε and μ are permitivity and permeability of the media, so we have

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}(\frac{1}{2}\vec{H}.\vec{B})$$

and $$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}(\frac{1}{2}\vec{E}.\vec{D})$$

Putting this in equation (6.4.14) and integrating over finite volume V we get

$$\int_V \vec{\nabla}.(\vec{E} \times \vec{H})\, dv = -\frac{d}{dt}\int_V [\vec{E}.\vec{D} + \vec{H}.\vec{B}]\, dv - \int_V \vec{J}.\vec{E}.dv \quad \text{..................(6.5.3)}$$

Applying Gauss's divergence theoreom to the left hand side of equation (6.5.3)

$$\int_S (\vec{E} \times \vec{H}).\hat{n}\, ds_o = -\frac{d}{dt}\int_V \frac{1}{2}(\vec{E}.\vec{D} + \vec{H}.\vec{B})\, dv - \int_V \vec{J}.\vec{E}.dv \quad \text{...............(6.5.4)}$$

where S is the surface bounding the volume V as,

$$-\frac{d}{dt}\int_V \frac{1}{2}(\vec{E}.\vec{D} + \vec{H}.\vec{B})\, dv = \int_V \vec{J}.\vec{E}.dv + \int_S (\vec{E} \times \vec{H}).\hat{n}\, ds_o \quad \text{..................(6.5.5)}$$

The term $\frac{1}{2}\vec{E}.\vec{D}$ is energy stored in electrical fields and the term $\frac{1}{2}(\vec{H}.\vec{B})$ is the energy stored in magnetic field. The left hand side of equation (6.5.5) points out the rate at which the electromagnetic energy stored in the volume V decreases with time. The rate of work done by the electromagnetic force on an infinetisimal charge $dq = \rho dV$ is given by

$$\frac{dw}{dt} = dq(\vec{E} \times \vec{\upsilon} \times \vec{B}).\vec{\upsilon}$$

$$= dq\,\vec{E}.\vec{\upsilon} = \vec{E}.\vec{\upsilon}(\rho dV) = \vec{E}.\vec{J}.dV$$

where υ is the velocity of the charge element and $\vec{J} = \rho\vec{\upsilon}$, So the term $\int \vec{E}.\vec{\upsilon}\, dV$ indicates the rate of doing work on the charge in the volume V by electromagnetic field, in other words it is Joule heat. The last term on the right hand side of equation (6.5.5) gives the rate which energy flows out of the bounding volume V.

The vector $\vec{S} = \vec{E} \times \vec{H}$ called Poynting vector has the unit of Joule m^{2} s^{-1} and so it can be described as the energy flowing out through unit area per unit time.

Poynting's Theorem can also be represented in differential form

$$\vec{\nabla}.\vec{S} + \frac{\partial W}{\partial t} = P_2 \qquad \text{..........(6.5.6)}$$

where $P_2 = J.E = \sigma.E^2$ as the Joule heat and $W = \frac{1}{2}(\mu H^2 + \varepsilon \vec{E})$ electromagnetic energy stored in per unit volume.

6.5.1 Time Average Value of The Poynting Vector

Now assuring the electric and magnetic fields is given by:

$$\vec{E} = \vec{E}_0 e^{jwt}$$
$$= (E_{Re} + j E_{IM})(\text{Cos}\omega t + j\text{Sin}\omega t)$$
$$= (E_{Re}\cos\omega t - E_{im}\text{Sin}\omega t)$$

and $\vec{B} = \vec{B}_o e^{jwt}$

$$= (B_{Re} + jB_{IM})(\cos\omega t + j\,\text{Sin}\omega t)$$

Now the Poynting vector $\vec{S}$ is given by,

$$\vec{S} = \frac{1}{\mu o}\,(\vec{E} \times \vec{B})$$
$$= \frac{1}{\mu o}\left[(E_{Re}\text{Cos}\omega t - E_{IM}\text{Sin}\omega t) \times (B_{Re}\cos\omega t - B_{IM}\sin\omega t)\right]$$
$$= \frac{1}{\mu o}\,[(E_{Re} \times B_{Re})\text{Cos}^2\omega t - (E_{Re} \times B_{IM})\text{Sin}\omega t\,\text{Cos}\omega t$$
$$-(E_{IM} \times B_{Re})\,\text{Cos}\omega t.\text{Sin}\omega t + (E_{IM} \times B_{IM})\text{Sin}^2\omega t]$$

Now taking the average of the above equation we get

$$\mu_o\langle\bar{S}\rangle = \langle(E_{Re} \times B_{Re})\rangle\langle\text{Cos}^2\omega t\rangle - \langle(E_{Re} \times B_{Im})\rangle\langle\text{Cos}\omega t\rangle\langle\text{Sin}\omega t\rangle$$
$$-(\langle E_{Im} \times B_{Re}\rangle\langle\text{Cos}\omega t\rangle\langle\text{Sin}\omega t\rangle + \langle E_{Im} \times B_{Im}\rangle\langle\text{Sin}^2\omega t\rangle$$

$$so\ \langle\bar{S}\rangle = \frac{1}{\mu_o}\left[\frac{1}{2}\langle E_{Re} \times B_{Re}\rangle - 0 - 0 + \frac{1}{2}\langle E_{IM} \times B_{IM}\rangle\right]$$
$$= \frac{1}{2\,\mu_o}\,[\langle E_{Re} \times B_{Re}\rangle + \langle E_{IM} \times B_{IM}\rangle]$$

But to compute the Poyting vector the simplest way to use a real form for both field $\vec{E}$ and $\vec{B}$ rather than complex exponential representation.

6.5.2 Energy of Electromagnetic Waves

A plane monochromatic EM waves propagating in Z direction ie: $\hat{K}$ direction is given by $\vec{E} = \hat{i}\, E_o \cos(kz-\omega t)$ and $\vec{B} = \hat{j}\, B_o \cos(kz-\omega t)$ ....(6.5.7)

where $B_o = \frac{E_o}{c}$

The total energy associated with the electromagnetic wave fields is

$$U = U_E + U_M = \frac{1}{2}\int_V \left(\frac{B^2}{\mu_o} + \varepsilon_o E^2\right) dV$$

as $|\vec{B}| = |\vec{E}/C|$ and c = $\sqrt{\varepsilon_0 \mu_o}$ The electric and magnetic energy contributions to the total energy are equal and electromagnetic energy density for a polarised wave is

$$U_{EM} = \varepsilon_o E^2 = \varepsilon_o E_o^2 \, Cos^2 (kz-\omega t)$$

The Poynting vector becomes

$$\vec{S} = \frac{1}{\mu_o} (\vec{E} \times \vec{B}) = c\varepsilon_o E_o^2 \, Cos^2(kz-\omega t)\hat{k}$$

$$= u_{EM} c\, \hat{k}$$

The time average density is defined as the average over one period T of the EM wave,

$$<u_{em}> = \frac{\varepsilon_o E_o^2}{T}\int_o^T Cos^2(kz-\omega t)dt$$

$$= \frac{\varepsilon_o E_o^2}{T}\frac{T}{2} = \frac{1}{2}\varepsilon_o E_o^2 = \frac{1}{2}\frac{B_o^2}{\mu_o} \quad \text{...(6.5.8)}$$

It follows that energy density of EM wave is proportinal to the square of the amplitude of the electric (or magnetic) field.

6.5.3 Momentum of Electromanetic Radiation

Due to The wave-particle duality of radiation, as stated in Quantum Mechanics, radiation or photonos travelling with speed c, the energy of each photon is given by

$\varepsilon = \hbar\omega = h\nu$

Momentum of a single photon is $\vec{P} = \hbar K = \frac{\varepsilon}{c}\hat{k}$

So for n photons per unit volume we can relate average Poynting vector to n$\in$, multiplied by velocity vector $c\hat{k}$

$$<S> = n\epsilon c\hat{k} = <u_{em}> c\,\hat{k}$$

Now $\vec{p}$ is defined as momentum of EM waves carried across a plane normal to propagation vector $\hat{k}$ per unit area per unit time $\vec{P} = \vec{S}/c$

when all the momentum of EM wave is absorbed in normal incidence, it exhibits a force per unit area equL to the normal incoming flux of radiations. The radiation pressare is

$P_{rad} = P.\hat{n} = S/c \Rightarrow P_{rad} <u_{EM}>$

In case of diffuse radiation ie. radiation bouncing around in all direction, the pressure is given by

$P_{rad} = <u_{EM}>/3$

6.6 Maxwell Stress Tensor

It is a symmetric second order tensor used in classical electromagnetics to represent the interaction between electromagnetic forces and mechanical momentum. A second rank tensor whose product with unit vector to a surface reveals the force per unit area transmitted across the surface by an electromagnetic field. Its easy to calculate the Lorentz force on the charge moving freely in homogeneous electromagnetic field, which is simple situation. In complex situation of interaction of particle and electromagnetic field, Maxwell stress tensor lays the way to use tensor arithmatic to find an answare to the problem at hand. Momentum conservation is rescued by the realization that fields themselves carry momentum, also its attributed energy.

As we know The Lorentz force on a moving charge particle is given by

$$\vec{F} = q\,(\vec{E} + \vec{\upsilon} \times \vec{B}) \quad \text{...(6.6.1)}$$

So the force per unit volume acting on charge density distribution ρ in a volume V

$$\vec{f} = \rho\,(\vec{E}+\vec{\upsilon}\times\vec{B}) \quad \text{.....(6.6.2)}$$

$$\Rightarrow \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} \quad \text{.....(6.6.3)}$$

From Maxwell's Electromagnetin equation

$$\rho = \varepsilon_o \vec{\nabla}.\,\vec{E} \quad \text{.....(6.6.4)}$$

$$\text{and } \vec{J} = \frac{1}{\mu_o}\,\vec{\nabla}\times\vec{B} - \varepsilon_o\,\frac{\partial\vec{E}}{\partial t} \quad \text{.....(6.6.5)}$$

Substituting ρ and $\vec{J}$ from equation (6.6.4) and (6.5.5) in equations (6.6.3) we get,

$$\vec{f} = \varepsilon_o(\vec{\nabla}.\vec{E})\vec{E} + \left(\frac{1}{\mu_o}\vec{\nabla}\times\vec{B} - \varepsilon_o\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B} \quad \text{.....(6.6.6)}$$

$$\text{Now, } \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t} \quad \text{.....(6.6.7)}$$

$$\text{or, } \frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial\vec{B}}{\partial t} \quad \text{.....(6.6.8)}$$

Also from Maxwell's third equation

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \quad \text{.....(6.6.9)}$$

Subsituting above in equation (6.6.8) we get

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\,(\vec{\nabla}\times\vec{E}) \quad \text{.....(6.6.10)}$$

From, equation (6.6.6) and (6.6.10), we get after rearraing

$$f = \varepsilon_0\,[(\vec{\nabla}.\vec{E})\,\vec{E}-\vec{E}\mathrm{x}(\vec{\nabla}\mathrm{x}\vec{E})] + \frac{1}{\mu_o}[(\vec{\nabla}\mathrm{x}\vec{E})\mathrm{xB}]-\varepsilon_o[\frac{\partial}{\partial t}(\vec{E}\times\vec{B})] \quad \text{.....(6.6.11)}$$

Introducing a term $(\vec{E}\times\vec{B})\vec{B}$ in the equation (6.6.11) to make it more symnetrical

$$\vec{f} = \varepsilon_o[(\vec{\nabla}.\vec{E})\,\vec{E}-\vec{E}\mathrm{x}(\vec{\nabla}\mathrm{x}\vec{E})] + \frac{1}{\mu_o}[(\vec{\nabla}.\vec{B})\,\mathrm{B}-\vec{B}\mathrm{x}\vec{\nabla}\mathrm{x}\vec{B}]-\varepsilon_o[\frac{\partial}{\partial t}(\vec{E}\mathrm{x}\vec{B})] \quad \text{....(6.6.12)}$$

From the property of gradient, we know that,

$$\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A} \text{ x } \vec{\nabla} \text{ x } \vec{B} + \vec{B} \text{ x } \vec{\nabla} \text{ x } \vec{A} + (\vec{A}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A}$$

$$\Rightarrow \vec{\nabla}(E^2) = \vec{\nabla}(\vec{E}.\vec{E}) = \vec{E} \text{ x} \vec{\nabla} \text{ x } \vec{E} + \vec{E} \text{ x } \vec{\nabla} \text{ x } \vec{E} + (\vec{E}.\vec{\nabla})\vec{E} + (\vec{E}.\vec{\nabla})\vec{E}$$

$$=2\vec{E} \text{ x } \vec{\nabla} \text{ x } \vec{E} + 2(\vec{E}.\vec{\nabla})\vec{E}$$

so, $$\vec{E} \text{ x} \vec{\nabla} \text{ x } \vec{E} = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}.\vec{\nabla})\vec{E} \qquad ....(6.6.13)$$

$$\vec{B} \times \vec{\nabla} \times \vec{B} = \frac{1}{2}\vec{\nabla}(B^2) - (\vec{B}.\vec{\nabla})\vec{B} \qquad ....(6.6.14)$$

Subsituting, (6.6.13) and (6.6.14) in equation (6.6.12)

$$\vec{f} = \varepsilon_o[(\vec{\nabla}.\vec{E}).\vec{E} - \frac{1}{2}\vec{\nabla}(E^2) + (\vec{E}.\vec{\nabla})\vec{E}] + \frac{1}{\mu_o}[(\vec{\nabla}.\vec{B})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2) + (\vec{B}.\vec{\nabla})\vec{B}]$$

$$- \varepsilon_o[\frac{\partial}{\partial t}(\vec{E} \text{ x } \vec{B})] \quad \text{or}$$

$$\vec{f} = [(\vec{\nabla}.\vec{E})\vec{E} + (\vec{E}.\vec{\nabla})\vec{E}] + \frac{1}{\mu_o}[(\vec{\nabla}.\vec{B})\vec{B} + (\vec{B}.\vec{\nabla})\vec{B}]$$

$$-[\frac{1}{2}\varepsilon_o\vec{\nabla}(E^2) + \frac{1}{2\mu_o}\vec{\nabla}(B^2)] - \varepsilon_o[\frac{\partial}{\partial t}(\vec{E} \text{ x} \vec{B})]$$

Equation (6.6.13) can be simplilfied with the introduction of Maxwells tress tensor

$$T_{ij} = \varepsilon o\,(E_i E_j - \frac{1}{2}\delta ij E^2) + \frac{1}{\mu_o}(B_i B_j - \frac{1}{2}\delta_{ij}B^2)................................(6.6.15)$$

The indices i' and j' refer to the co-ordinates x, y ans z, so the stress tensor has a total nine components (T_{xx}, T_{xy}, T_{xz}, T_{yx}, T_{yy}, T_{yz}, T_{zx}, T_{zy},T_{zz})

Thus,

$$T_{xx} = \frac{1}{2}\varepsilon o\,(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2}\frac{1}{\mu_o}(B_x^2 - B_y^2 - B_z^2)...............(6.6.16)$$

$$T_{yy} = \frac{1}{2}\varepsilon o\,(E_y^2 - E_x^2 - E_z^2) + \frac{1}{2}\frac{1}{\mu_o}(B_y^2 - B_x^2 - B_z^2).............(6.6.17)$$

$$T_{zz} = \frac{1}{2}\varepsilon o\,(E_Z^2 - E_x^2 - E_y^2) + \frac{1}{2}\frac{1}{\mu_o}(B_z^2 - B_y^2 - B_x^2)...............(6.6.18)$$

and, $$T_{xy} = T_{yx} = \varepsilon o\, E_x E_y + \frac{1}{\mu_o} B_x B_y \;.....................................(6.6.19)$$

$$T_{yz} = T_{zy} = \varepsilon o\, E_y\, E_z + \frac{1}{\mu_o} B_y\, B_z \quad \text{.......................................(6.6.20)}$$

$$Tzx = T_{xz} = \varepsilon o\, E_z\, E_x + \frac{1}{\mu_o} B_z\, B_x \quad \text{.......................................(6.6.21)}$$

T_{ij} is represented as a tensor by $\overleftrightarrow{T}$ a rank 2- tensor. It is represented by a 2-dimensional, (3x3) matrix

$$\overleftrightarrow{T} = \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix} \quad \text{...(6.6.22)}$$

We can form the dot product of $\overleftrightarrow{T}$ with a vector $\vec{a}$

$$(\vec{a}.\overleftrightarrow{T})\, j = \Sigma_{x,y,z}\, (a_j\, T_{ij}) \quad \text{...(6.6.23)}$$

the out-coming object, which has one remaining index, is itself a vector. Now if we take the divergence of $\overleftrightarrow{T}$ has as its jth component.

$$(\vec{\nabla}.\overleftrightarrow{T})\, j = \varepsilon o\, [(\vec{\nabla}.\vec{E})\, Ej + (\vec{E}.\vec{\nabla})\, Ej] + \frac{1}{\mu_o}\, [(\vec{\nabla}.\vec{B})\, Bj + (\vec{B}.\vec{\nabla})\, Bj]$$

$$-[\frac{1}{2}\varepsilon o\, \nabla j\, (E^2) + \frac{1}{2\,\mu_o} \nabla j\, (B^2)] \quad \text{...(6.6.24)}$$

So the force per unit volume in equation (6.6.24) take the form,

$$\vec{f} = (\vec{\nabla}.\overleftrightarrow{T}) - \varepsilon o\, \mu_o \frac{\partial \vec{S}}{\partial t} \quad \text{..(6.6.25)}$$

So The total force on the charges in volume V is given by,

$$\vec{F} = \int_V f dv = \int_V (\vec{\nabla}.\overleftrightarrow{T})\, dv - \varepsilon o\, \mu_o \int \frac{\partial \vec{S}}{\partial t} dv \quad \text{...................................(6.6.26)}$$

or $$\vec{F} = \int f dv = \oint \overleftrightarrow{T}.\vec{ds}_o - \varepsilon o\, \mu_o \int \frac{\partial \vec{S}}{\partial t} dv \quad \text{..(6.6.27)}$$

Here S_o represents the surface. In the static case, $\int \frac{\partial S}{\partial t} dv$ is to be dropped.

Physical signticance of $\overleftrightarrow{T}$ is the force per unit area (or stress) acting on the surface. Here Tij is the force (per unit area) in the ith direction acting on an element of surface alligned in the j th direction. The diagonal elements represent pressures , and off diagonal elements are shears.

6.6.1 Conservation of Momentum

According to Newton's second law the force on an object is equal to its momentum

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} \quad \text{.......(6.6.28)}$$

So equation (6.6.27) can be writen in the form given below

$$\vec{F} = \frac{d(\vec{P}_{mech})}{dt} = \varepsilon_0 \mu_o \int \frac{\partial \vec{S}}{\partial t} dv + \oint \overleftrightarrow{T}.\vec{d}s_o \quad \text{.......(6.6.29)}$$

where $\vec{P}_{mech}$ is the total mechanical momentum of the particles in volume V. This expression in equation (6.6.29) is similar to the representation of Poynting theorem. The first integral represents momentum stored in the electromagnetic fields themselves

$$g_{em} = \varepsilon_0 \mu_o \int \vec{S}\, dv \quad \text{.......(6.6.30)}$$

while the second integral is the momentum per unit time flowing in throgh the surface equation (6.6.29) is the general statement of conversation of momentum in electrodynamics. Any increase in the total momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields when V encompass all space then, no momentum flows in or out, and $P_{mech} + g_{em}$ is constant.

If the mechanical momentum is V is not changing ie in region of empty space, then

$$\int \frac{\partial g_{en}}{dt} dv = \oint \overleftrightarrow{T}.\vec{d}s_o = \int \vec{\nabla}.\overleftrightarrow{T} dv$$

and hence $\frac{\partial g_{en}}{dt} = \vec{\nabla}.\overleftrightarrow{T}$ ....(6.6.31)

This is the "continuity equation" for electromagnetic momentum, with g_{em} (momentum density) in the role of ρ (charge density) and $\overleftrightarrow{T}$ playing the role of $\vec{J}$; it expresses the total local conservation of field momentum. But in general charges and fields exchange momentum and only the total is conserved.

Here we note that $\vec{S}$ plays the energy per unit area per unit time transported bythe fields, while $\mu_o \varepsilon o \vec{S}$ is the momentum per unit volume stored in those fields. Similarly $\vec{T}$ is the electromagnetic stress acting on thesurface and- $\vec{T}$ represent's the flow of momentum ie momentum current density, carried by the fields.

6.6.2 Angular Momentum

The angular momentum of EM wave is a vector that expresses the amount of dynamical rotation present in the electromagnetic while travelling approximately in a straight line. The beam of light can also be taking, the two distinct forms of rotation of light beam are its polarization and its wave front shape. Two forms of rotation are identified as light spin angular momentum.

Now the energy density of electromagnetic fields carry energy,

$$u = \frac{1}{2} (\varepsilon o\ E^2 + \frac{1}{\mu_o} B^2) \qquad \text{.......(6.6.32)}$$

and momentum,

$$\vec{g}_{em} = \varepsilon o\ (\vec{E} \, x \vec{B}) \qquad \text{.......(6.6.33)}$$

The Angular momentum

$$\vec{L} = \vec{r} \; x \; \vec{g}_{em} = \varepsilon o\ [\vec{r} \; x (\vec{E} \, x \vec{B})] \qquad \text{.......(6.6.34)}$$

In case of static fields, it can have angular momentum as long as $\vec{E} \; x \; \vec{B}$ is non zero and it is only when these field contributions are incorporated that the conversation laws are prevailed.

6.7 Potencial Formulation and Gauge Transformations

In Maxwell's theory, the basic field variable are the strenghts of electric and magnetic fields which may describe in terms of auxillary variable like scalar and vector potentials. The gauge (theory) transformations in this theory consists of certain changes in value of these potentials that do not yield in a change of the value of electric and magnetic fields. Thus the invariance is preserved as we look forward to the formulation of modern theory of electrodynamics.

In electrostactics we know $\vec{\nabla} \; x \; \vec{E} = 0$ it enables one to write $\vec{E} = \; \vec{\nabla}\ \phi$, where ϕ is a scalar function. In electrodynamics $\vec{\nabla} \; x \; \vec{E} \neq 0$ But $\vec{\nabla}. \vec{B}. = 0$. So it demands for certain generalisation time dependent solution of the problems.

So we can write variable $\vec{B}$ as

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

where is $\vec{A}$ vector potencial.

$$\text{Now} \Rightarrow \vec{\nabla} \times [\vec{E} + \frac{\partial \vec{A}}{\partial t}] = 0 \quad \text{......(6.7.1)}$$

Above equation (6.7.1) can be transformed with the introduction of gradient of a scalar function ϕ to the expression $(\vec{E} + \frac{\partial \vec{A}}{\partial t})$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{......(6.7.2)}$$

In static case $\frac{\partial \vec{A}}{\partial t} = 0$, so, $\vec{E} = -\vec{\nabla}\phi$

from Gauss's law $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon o}$ we have,

$$\nabla^2\phi - \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\rho / \varepsilon_0 \quad \text{......(6.7.3)}$$

substituting equation (6.7.2) in modified Ampere's law in electrodynamincs,

$$\vec{\nabla} \times \vec{B}) = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$(\vec{\nabla} \times \vec{\nabla} \times \vec{A}) = \mu_o \vec{J} - \mu_o \varepsilon o\ \vec{\nabla}(\frac{\partial \phi}{\partial t}) - \mu_o \varepsilon o \frac{\partial^2 \vec{A}}{\partial t^2} \quad \text{......(6.7.4)}$$

from vector identity $\vec{\nabla} \times (\vec{\nabla}.\vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}$

we have from equation on (6.7.4)

$$(\vec{\nabla}^2 \vec{A} - \mu_o \varepsilon o\ \frac{\partial^2 \vec{A}}{\partial t^2}) - \vec{\nabla}(\vec{\nabla}.\vec{A} + \mu_o \varepsilon o \frac{\partial \phi}{\partial t}) = -\mu_o \vec{J} \quad \text{......(6.7.5)}$$

So from equation (6.7.3) and (6.7.5) carry all the informations in Maxwell's equations. We conclude that potential formulation of Maxwell's electromagnetic formulation reduce the six variable of $\vec{E}$ and $\vec{B}$ (three of each) to four variables are three values of vector potencial $\vec{A}$ and one value to sealar function ϕ.

Equation (6.7.2) and the equation $\vec{B} = \vec{\nabla} \times \vec{A}$ do not uniqely define the potentials. Let us introduce two sets of potential $(\phi, \vec{A})$ and $(\phi', \vec{A}')$, gives the same electirc and magnetic fields, so writing $\vec{A} = \vec{A} + \vec{\alpha}$ and $\phi' = + \beta$,

We have,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\alpha}) \text{ or } \vec{\nabla} \times \vec{\alpha} = O$$

or $$\vec{\alpha} = \vec{\nabla} S$$

where S is scalar

Again, $$\vec{E} = \vec{-}(\nabla\phi + \frac{\partial \vec{A}}{\partial t}) = -(\nabla\phi' + \frac{\partial \vec{A}'}{\partial t})$$

$$\vec{\nabla} B + \frac{\partial \alpha}{\partial t} = O$$

or $$\vec{\nabla}(B + \frac{\partial S}{\partial t}) = O$$

So the term $(B + \frac{\partial S}{\partial t})$ is independant of position co-ordinates, it is only function of time, taking it as g(t), thus

$$B = -\frac{\partial S}{\partial t} + g(t)$$

or $$B = -\frac{\partial P}{\partial t}$$

where $P = S - \int_o^t g(t')dt'$

The function P can replace S in the defination of

$$\vec{\alpha}. \text{ since} = \vec{\nabla} P = \vec{\nabla} S, \text{ so}$$

$$A' = \vec{A} + \vec{\nabla} P \quad \text{...(6.7.6)}$$

and $$\phi' = \phi - \frac{\partial P}{\partial t} \quad \text{...(6.7.7)}$$

Thus, we observe that addition of $\vec{\nabla} P$ to $\vec{A}$ and the subtraction of $\frac{\partial P}{\partial t}$ from ϕ do not alter the $\vec{E}$ and $\vec{B}$. these changes in ϕ and $\vec{A}$ are called gauge transformations.

We can also choose set of potential $\vec{A}$, ϕ, such that,

$$\vec{\nabla}.\vec{A}+\mu_o\varepsilon_o\frac{\partial^2\vec{A}}{\partial t^2}=O \quad \text{...(6.7.8)}$$

This choice is called Lorentz gauge under this transformation, equation (6.7.3) and (6.7.5) becomes,

$$\nabla^2\vec{A}-\mu_o\varepsilon_o\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_o\vec{J} \quad \text{..(6.7.9)}$$

$$\nabla^2\phi-\mu_o\varepsilon_o\frac{\partial^2\phi}{\partial t^2}=-\rho/\varepsilon_o \text{..(6.7.10)}$$

We can choose another set of gauge called coulomb gauge, where $\vec{\nabla}.\vec{A}=O$, Equation (6.7.3) and (6.7.5) become,

$$\nabla^2\vec{A}-\mu_o\varepsilon_o\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_oJ+\mu_o\varepsilon_o\frac{\partial\phi}{\partial t} \text{.................................(6.7.11)}$$

and, $\nabla^2\phi=-\rho/\varepsilon_o$ ....(6.7.12)

Equation (6.7.12) can easyly be solved to find ϕ, as in electrostatics

$$\phi(r,t)=\frac{1}{4\pi\varepsilon_o}\int\frac{(\vec{r}',t)}{1\,\vec{r}-\vec{r}'1}dv$$

6.8 Boundary Conditions

We can use Maxwell's equations to derive the boundry condition on the magnetic field across a surface. Consider a "pillbox" across the surface taking Maxwell's equation

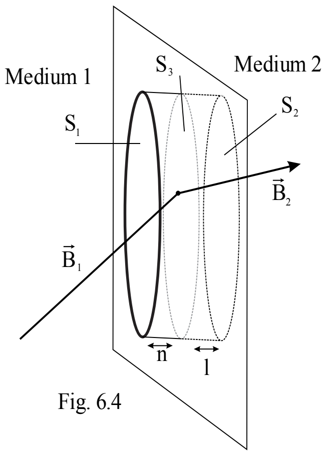


Fig. 6.4

$$\vec{\nabla}.\vec{B}=O \text{...(6.8.1)}$$

intergate over the volume of the pillbox, apply Gauss's the orem :

$$\int_V\vec{\nabla}.\vec{B}\,dv=\oint_S\vec{B}.\,\vec{ds}=O \text{.......................................(6.8.2)}$$

where V is the volume of the pillbox, and S its surface. We can break the integral over the surface into three part's over the flat ends (S_1 and S_2) and over the curved wall (S_3) [Fig.6.4]

$$\int_{S_1}\vec{B}.\,\vec{ds}+\int_{S_2}\vec{B}.\,\vec{ds}+\int_{S_3}\vec{B}.\,\vec{ds}=O \quad\text{.......................(6.8.3)}$$

In the limit that the lenght of the pillbox approaches zero the integral over the curved surface also approaches zero. It each end has small area "A" then equation (6.8.3) becomes

$$-B_{in}A+B_{2n}A=O \quad\text{...............................(6.8.4)}$$

$$B_{in}=B_{2n} \quad\text{...(6.8.5)}$$

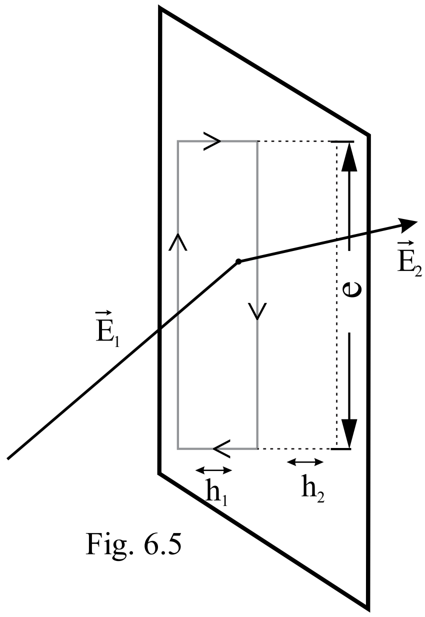


Fig. 6.5

it impiles the normal component of the magnetic field $\vec{B}$. must be continuous across the surface.

Boundary condition 2: Tangential Component of $\vec{E}$ consider a loop spanning the surface (Fig. 6.5)

$$\text{Maxwell equation : } \vec{\nabla}\times\vec{E}-\frac{\partial\vec{B}}{\partial t} \quad\text{..........(6.8.6)}$$

Integrate over the surface bounded by the loop and apply stoke's theorem to get

$$\int_S\vec{\nabla}\times\vec{E}.\,\vec{ds}=\oint_C\vec{E}.\,\vec{dl}=\frac{\partial}{\partial t}\int_S\vec{B}.\,\vec{ds} \quad\text{.............(6.8.7)}$$

Now take the limit, in which the width of the loop becomes zero. The contributions to integral around the loop C from narrow ends become zero; as does integral of the magnetic field across the area bounded by the loop, so from equation (6.8.7)

$$E_{1t}\ell-E_{2t}\ell=O \quad\text{....................................(6.8.8)}$$

$$\text{so, } E_{1t}=E_{2t} \quad\text{..(6.8.9)}$$

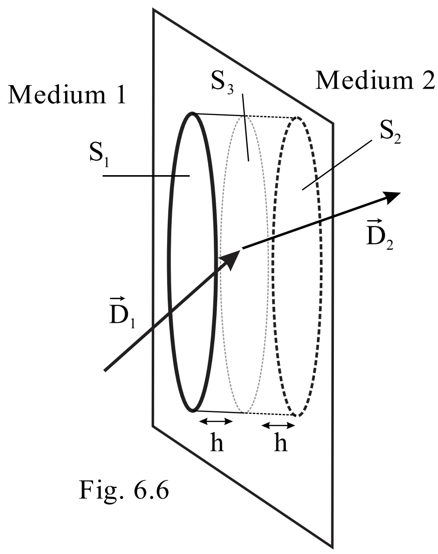


Fig. 6.6

Hence the tangential component of the electric field is continuous across the boundary.

Boundary Condition 3 : Normal Component of $\vec{D}$, consider a pillbox crossing the boundary (Fig. 6.6)

From Maxwell's equation,

$$\vec{\nabla}.\vec{D} = \rho \text{ ....(6.8.10)}$$

Integrating over the volume of pillbox, apply Gauss's Theorem

$$\int \vec{\nabla}.\ \vec{D}\ dv = \oint \vec{D}.d\vec{s} = \int \rho dv \text{ ....(6.8.11)}$$

Assuming the height of the pillbox to zero, and surface charge density s3 area of the pillbox being small A, then

$$- D_{1n} A + D_{2n}A = \sigma_s A$$

or $\quad D_{2n} - D_{1n} = \sigma_s$ ....(6.8.12)

When surface charge density is zero. The normal component of $\vec{D}$ is continuous across the boundary, However it is not true for the normal component of $\vec{E}$ unless the two materials have identical (permittvities). From continuitiy equation,

$$\vec{\nabla}.\vec{J} = \frac{\partial \rho}{\partial t} \text{ ....(6.8.13)}$$

Integrating equation (6.8.13) over the volume pullbox having approximately zero height, we get from (6.8.10)

$$J_{1n} - J_{2n} = -\frac{\partial \sigma_s}{\partial t} \text{ ....(6.8.14)}$$

For monochromatic electro magnetic wave, s_s will vary as e^{-jwt} then

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \sigma_s \text{ ....(6.8.15)}$$

and, $\sigma_1 E_{1n} - \sigma_2 E_{2n} = jw\sigma_s$...(6.8.16)

Now consider the following cases

It σ_s = O. from equation (6.8.15) and (6.8.16)

i) $\dfrac{\varepsilon_1}{\sigma_{c1}} = \dfrac{\varepsilon_2}{\sigma_{c2}}$ ....(6.8.17)

Which can be satisfield for properly chosen materials,

ii) If $\sigma_s \neq o$, eleminating σ_s from equation (6.8.15) and (6.8.16),

$$[\varepsilon_1 + \frac{j\sigma_{c1}}{\omega}]\, E_{1n} = [\varepsilon_2 + j\,\frac{\sigma_{c2}}{\omega}]\, E_{2n} \quad \text{......(6.8.18)}$$

iii) It $\sigma_{c2} = \infty$, then $E_{2n} = O$ since the electric field inside a perfect conductor

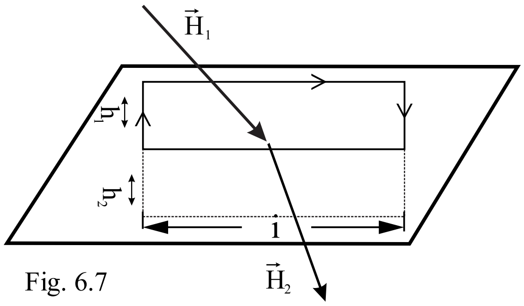


Fig. 6.7

must be zero. From equation (6.8.15), we have $E_{1n} = \frac{\sigma_s}{\varepsilon_1}$ as $D_{1n} = \sigma_s$ when electromagnetic waves pass into a conductor, the field amplitudess fall exponentially with a decay lenght given by the skin depth

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \quad \text{......(6.8.19)}$$

As conductivity increases, the skin depth gets smaller. Since both static and oscillating electric fields vanish within a good conductor, the boundary condition is given by:

$$\begin{aligned} &E_{1t} \approx O \qquad E_{2t} = O \\ &D_{1n} \approx \sigma_s \qquad D_{2n} \approx O \end{aligned} \quad \text{......(6.8.20)}$$

Tangential Component of $\vec{H}$

consider a loop across the boundary (Fig. 6.7)

From Maxwell's equations $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$,we intergrate over the surface bounded by the loop, and apply Stoke's theorem to obtain

$$\int_S \vec{\nabla} \times \vec{H}.\, \vec{ds} = \oint_C \vec{H}.\, \vec{dl} = \int_S \vec{J}.\, \vec{ds} + \frac{\partial}{\partial t}\int \vec{D}.\, \vec{ds} \quad \text{..............................(6.8.21)}$$

As before taking the limit where the lengh of the narrow edges of the loop become zero then we have,

$$H_{1t}l - H_{2t}l = J_{s\wedge} \quad \text{..(6.8.22)}$$

Where $J_{s\wedge}$ represents a surface current density perpendicular to the direction of the tangential component of $\vec{H}$ that is being matched.

6.9 Wave Equation

Formulation of complete and symmetric theories of electricity and magnetism, together with Lorentz force law, by Maxwell, have culminated in the prediction of wave theory of light identified and discovered as electrimagnetic wave, which travells with the speed C.

Let us assume that the medium is linear permittivity e, the permeability μ and electrical conductivity are constant. The wave equation for magnetic intersity is obtained by taking curl of $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = \vec{\nabla} \times \vec{J} + \vec{\nabla} \times \frac{\partial \vec{D}}{\partial t} \quad \text{...(6.9.1)}$$

As the current density, $\vec{J} = \sigma_c \vec{E}$ and electric displacement, $\vec{D} = \varepsilon \vec{E}$, so from equation (6.9.1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = \sigma_c \vec{\nabla} \times \vec{E} + \varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \quad \text{.......................................(6.9.2)}$$

Putting the value of $\vec{\nabla} \times \vec{E}$ from Maxwell's equation and given $\vec{B} = \mu \vec{H}$ simplitying equation (6.9.2)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = -\mu\sigma_c \frac{\partial \vec{H}}{\partial t} - \mu\, \varepsilon \frac{\partial^2 \vec{H}}{\partial t} \quad \text{..(6.9.3)}$$

using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 \vec{A}$

From equation (6.9.3) then

$$\vec{\nabla}\,\vec{\nabla}.\ \vec{H} - \nabla^2\vec{H} = -\,\sigma_c\mu\,\frac{\partial\vec{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}\ldots\ldots(6.9.4)$$

using, $\vec{\nabla}.\,\vec{H} = \frac{1}{\mu}\,\vec{\nabla}.\vec{B} = O$, Equation (6.9.4) becomes

$$\nabla^2\,\vec{H} = -\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} - \sigma_c\mu\frac{\partial\vec{H}}{\partial t} = O\ldots\ldots(6.9.5)$$

Equation (6.9.5) is the wave equation.

Again from Maxwell's equation, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ we can obtain,

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = -\,\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}\ldots\ldots(6.9.6)$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\,\mu\sigma_c\frac{\partial\vec{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}\ldots\ldots(6.9.7)$$

or, $$\vec{\nabla}\,\vec{\nabla}.\vec{E} - \nabla^2\vec{E} = -\,\sigma_c\mu\frac{\partial\vec{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}\ldots\ldots(6.9.8)$$

If the medium contains no charge $\rho = O$, so that $\vec{\nabla}.\vec{E} = \frac{1}{\varepsilon}\vec{\nabla}.\,\vec{D} = O$, so equation (6.9.8) becomes,

$$\nabla^2 E - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} - \sigma_c\mu\frac{\partial\vec{E}}{\partial t} = O\ldots\ldots(6.9.9)$$

Which is a wave equation.

6.10 Propagtion of EM Waves in Free Space ie $\tau = 0$, and $\sigma = 0$, J = 0

An EM wave unlike mechanical waves which requires the presence of material media to transport energy from one location to another space, carries the energy throngh a vacuum at a spaced of C = 3 x 10^8m/s which will be proved here from all the electromagnetic equations of Maxwell.

In free space, Maxwell's equations become

$$\vec{\nabla}.\vec{E} = 0 \quad \text{.....(6.10.1)}$$

$$\vec{\nabla}.\vec{H} = 0 \quad \text{.....(6.10.2)}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \text{.....(6.10.3)}$$

$$\vec{\nabla} \times \vec{H} = -\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{.....(6.10.4)}$$

Taking the curl of equation (6.10.3) we obtain,

$$\vec{\nabla}\,(\vec{\nabla}.\vec{E}) - \nabla^2 E = \mu_0 \frac{\partial}{\partial t}\,(\vec{\nabla} \times \vec{H})$$

or $$\vec{\nabla}^2\vec{E} - \varepsilon_0\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = O \quad \text{.....(6.10.5)}$$

Similary taking curl of equation (6.10.4) and using (6.10.3) we get,

$$\nabla^2\vec{H} - \varepsilon_0\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{.....(6.10.6)}$$

Thus it appears that both $\vec{E}$ and $\vec{H}$ satifiys the well known wave equation -

$$\nabla^2\phi - \frac{1}{c}\frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{.....(6.10.7)}$$

So the velocity of propagetion of EM wave is

$$C = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8 \text{m/s}$$

Which is exactly the speed of light in free space, there is a corelation can be drawn that light is a form of EM waves.

Let us seek a simple solutions concerning $\vec{E}$ or $\vec{H}$

$$\vec{E}\,(\vec{r}, t) = \vec{E}_0\,\varepsilon^{j(\vec{\kappa}.\vec{r}-\omega)} \text{ and } \vec{H}\,(\vec{r}, t) = \vec{H}_0\varepsilon^{j(\vec{\kappa}.\vec{r}-\omega t)} \quad \text{.....(6.10.8)}$$

Where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes. Which are constants in space and time $\vec{\kappa}$ is the were vector determining the direction of prepagation of wave, $\vec{\kappa}$ is defined as

$$\vec{\kappa} = \frac{2\,\hat{h}}{\lambda}\hat{n} = \frac{\omega}{c}\hat{n} \quad \text{.......(6.10.9)}$$

Where $\hat{n}$ is the unit vector along the direction of propagation.

Therefore, $\nabla^2 \vec{E} = -\kappa^2 \vec{E}$ and $\frac{d^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}$ ....(6.10.10)

subsituting equation (6.10.10) in equation (6.10.5) we get

$$\kappa^2 \vec{E} = -\mu_0\varepsilon_0\,\omega^2 E$$

$$\kappa^2 = \mu_0\varepsilon_0\,\omega^2$$

$$\frac{\omega^2}{\kappa^2} = \frac{1}{\mu_0\varepsilon_0\,\omega^2}$$

or $\quad v^2 = \frac{1}{\mu_0\varepsilon_0\,\omega^2}$ ....(6.10.11)

Now plugging in the value of μ_0 = 4 x 10^{-7} H/m and ε_0 = 8.85 x 10^{-12} F/m, we get v = $3x10^{10}$ m/s = speed of light(c)

Relative diections of $\vec{E}$ $\vec{H}$ and $\vec{\kappa}$:

From equations (6.10.8), (6.10.1) and (6.10.2) it can be show that.

$$\vec{\kappa}, \vec{E} = 0 \text{ and } \vec{\kappa}.\vec{H} = 0 \quad \text{.......(6.10.12)}$$

so both $\vec{E}$ and $\vec{H}$ are perpendicular to the propagation vector $\vec{K}$, which implies the transverse characteristic of EM wave or light wave.

From equation (6.10.8), (6.10.3) and (6.10.4) it can be shown that

$$j\,\vec{\kappa}\,x\,\vec{E} = -\mu_0\,(-j\omega\vec{H}) \text{ or } \vec{\kappa}\,x\,\vec{E} = \mu_0\omega\vec{H} \quad \text{.......(6.10.13)}$$

and $\quad j\,\vec{\kappa}\,x\,\vec{E} = -\varepsilon_0\,(-j\omega\vec{E})$ or $\vec{\kappa}\,x\,\vec{H} = \varepsilon_0\omega\vec{E}$ ....(6.10.14)

Equation (6.10.13) shows that $\vec{H}$ is both perpendicular to both $\vec{K}$ and $\vec{E}$. Equation (6.10.14) shows that $\vec{E}$ is both perpendicular to both $\vec{K}$ and $\vec{H}$. Hence field vectors $\vec{E}$ and $\vec{H}$ are mutually perpendicular and also both are perpendicular to the direction of propagation vector $\vec{K}$ As $K^2 = \varepsilon_0\mu_o\omega^2$, thus in vaccum, K is real quantity, it proves that both $\vec{E}$ and $\vec{H}$ are in phase.

Wave Impedance :

The ratio of the absolute value $\vec{E}$ and $\vec{H}$ is defined as wave impendance.

$$Z_0 = \left|\frac{\vec{E}}{\vec{H}}\right| = \frac{\mu_0\omega}{\kappa} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

The value of Z_0 comes around 376.6 Ω.

6.11 Plane EM Waves in an Isotropic Dietectric Medium

Let us consider a linear homogeneous and isotropic dietectric medium where $\rho = 0$, Maxwell's equation then becomes

$$\vec{\nabla}.\,\vec{E} = 0 \qquad (6.11.1)$$

$$\vec{\nabla}.\,\vec{H} = 0 \qquad (6.11.2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\mu\,\partial \vec{H}}{\partial t} \qquad (6.11.3)$$

$$\vec{\nabla} \times \vec{H} = -\,\epsilon\,\frac{\partial \vec{E}}{\partial t} \qquad (6.11.4)$$

Taking curl of equation (6.11.3) and using (6.11.4), we obtain

$$\vec{\nabla}\,(\vec{\nabla}.\vec{E}) - \nabla^2 E = -\mu\,\frac{\partial}{\partial t}\,(\vec{\nabla} \times \vec{H}) = -\frac{\epsilon\mu\delta^2\vec{E}}{\partial t^2}$$

As $\vec{\nabla}.\,\vec{E} = 0$ for no charge present in the dielectric

$$\nabla^2\vec{E} - \mu\varepsilon\,\frac{\partial^2\vec{H}}{\partial t^2} = 0 \qquad (6.11.5)$$

In the same way, taking curl of equation (6.11.4) and $\vec{\nabla}.\,\vec{H} = 0$, we get

$$\nabla^2\vec{H} - \mu\varepsilon\,\frac{\partial^2\vec{H}}{\partial t} = 0 \qquad (6.11.6)$$

So both $\vec{E}$ and $\vec{H}$ follow the standrad differential wave equation. So we get the velocity of electromagnetic waves in dielectric medium,

$$v = \frac{1}{\sqrt{\mu\,\varepsilon}} = \sqrt{\frac{1}{(\varepsilon_0\mu_0\varepsilon_r\mu_r)}} = \sqrt{\frac{C}{\varepsilon_r\mu_r}} \text{ ....(6.11.7)}$$

Where $C = \sqrt{\frac{1}{\varepsilon_0\,\mu_0}}$ is the speed of EM waves in free space.

ε_r is the permittivity or deelectric constant K.

μ_r is the relative permeability of the medium For a nonmagnetic dielectric medium $\mu_r = 1$ so,

$$v = \frac{C}{\sqrt{k}}$$

Solutions of the wave equations (6.11.5) and (6.11.6) are given by

$$\vec{E}\,(\vec{r}, t) = \vec{E}_0\, e^{J(\vec{k}.\vec{r} - \omega t)}$$

$$\vec{H}\,(\vec{r}, t) = \vec{H}_0\, e^{J(\vec{k}.\vec{r} - \omega t)} \text{ ....(6.11.8)}$$

Where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes, which are constants in both space and time and, wave vector is given by

$$\vec{\kappa} = \kappa\,\hat{n} = \frac{2\pi}{\lambda}\,\hat{n} = \frac{\omega}{c}\,\hat{n} \text{ ....(6.11.10)}$$

It shows that both $\vec{E}$ and $\vec{H}$ are prependicular to the direction of propagation vector $\vec{\kappa}$, which reveals that nature of electromagnetic waves.

subtituting the solution given in equation (6.11.8) to (6.11.4) we obtain

$$\vec{K} \text{ x } \vec{E} = \mu\omega\vec{H} \text{ ....(6.11.11)}$$

$$\vec{K} \text{ x } \vec{H} = -\varepsilon\omega\vec{E} \text{ ....(6.11.12)}$$

From equations (6.11.11) and (6.11.12) we can conclude that both $\vec{E}$ and $\vec{H}$ are prependicular to each other and also both of them are perpendicular to the direction of propagation vector $\vec{K}$.

From equation (6.11.8) and from the wave equation we can get

$$K^2 = \varepsilon\mu\omega^2 \qquad \text{.......(6.11.13)}$$

It states that in this stated dielectric properties wave vector $\vec{K}$ is a real quanlity and from equations (6.11.11) and (6.11.12) it can be shown that both $\vec{E}$ and $\vec{H}$ are in phase.

Wave impedance is found to be

$$Z = \left|\frac{\vec{E}}{\vec{H}}\right| = \frac{\mu\omega}{\kappa} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{.......(6.11.14)}$$

or $$Z = Z_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{Z_0}{\sqrt{\kappa}}$$

Poynting Vector : Poynting Vector $\vec{S}$ is given by

$$\vec{S} = \vec{E} \times \vec{H} \qquad \text{.......(6.11.15)}$$

sustituting the volume of $\vec{H}$ from equation (6.11.11) in equation (6.11.15) we get,

$$\vec{S} = \frac{1}{\mu\omega}\,\vec{E} \times (\vec{K} \times \vec{E})$$

$$= \frac{1}{\mu\omega}\,[\vec{K}\,(\vec{E}.\,\vec{E}) - \vec{E}\,(\vec{E}.\,\vec{K})]$$

$$= \frac{E^2}{\mu\omega}\,\vec{K} \qquad \text{.......(6.11.16)}$$

So, $$\vec{S} = \frac{E^2}{\mu\omega}\,\vec{K}$$

Equation (6.11.16) shows that energy flows in the direction of propagation vector $\vec{K}$, we can write the magnitudes as

$$K\,E = \mu\omega H$$

or $$\sqrt{\varepsilon}\,E = \sqrt{\mu}\,H$$

or $$\frac{1}{2}\epsilon E^2 = \frac{1}{2}\mu H^2 \qquad (6.11.17)$$

Therefore energy is equally shared between electric and magnetic field in linear isotropic and homogeneous dietecric medium.

Total electromagnetic energy density is

$$u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2 = \epsilon E^2 = \mu H^2 \qquad (6.11.18)$$

we can also write Poynting vector as

$$S = \frac{u}{\epsilon\mu\omega}\kappa\hat{n} = uv\hat{n} \qquad (6.11.19)$$

Where $\hat{n}$ is the unit vector in the direction of propagation. So the magnitude of Poynting vector is velocity of the wave multiplied by the energy density and propagate in harmony with the electric and magnetic field.

6.12 Reffection and Refractron at the plane interface of two Dietectrics: Normal Incidence

The interface separating the two different dielectrics is taken to be the X-Y plane at Z = 0. A plane monochromatic EM wave of angular frequency w incident normaly at the interface from medium with electric field vector along the X-axis. The wave will be partly reflected into the medium 1 and partly transmitted into the medium 2. The frequency of the reflected and transmitted wave will be same as the boundary condition must be the same at all times. The incident and transmitted waves will move along positive X-diretion. While reflected wave along negative X-axis.

The incident reflected and transmitted wave have electric and magnetic field vector are as follows:

Incident wave : $$\vec{E}_1 = \hat{i}\, E_{1x} e^{j(k_1 z - \omega t)}$$

$$\vec{H}_1 = \hat{J}\, H_{1y} e^{j(k_2 z \omega t)}$$

Reflected wave : $$\vec{E}_1' = -\hat{c}\, E'_{1x} e^{-(k_1 z + \omega t)}$$

$$\vec{H}_1' = -\hat{i}\,H_{1y}'\,e^{-j(k_1 z + \omega t)}$$

Transmitted wave : $\vec{E}_2 = \hat{i}\,E_{2x}\,e^{j(k_1 z - \omega t)}$

$$\vec{H}_2 = \hat{j}\,H_{2y}\,e^{j(k_2 z - \omega t)}$$

Taking the magnetic permeability of the medium to be μ_0, we can write magnetic field vectors as follows :

$$H_{1y} = \frac{n_1}{\mu_0 c}E_{1x},\; H_{2y} = \frac{n_2}{\mu_0 c}E_{2x} \text{ and } H_{1y}' = \frac{n_1}{\mu_0 c}E_{1x}'$$

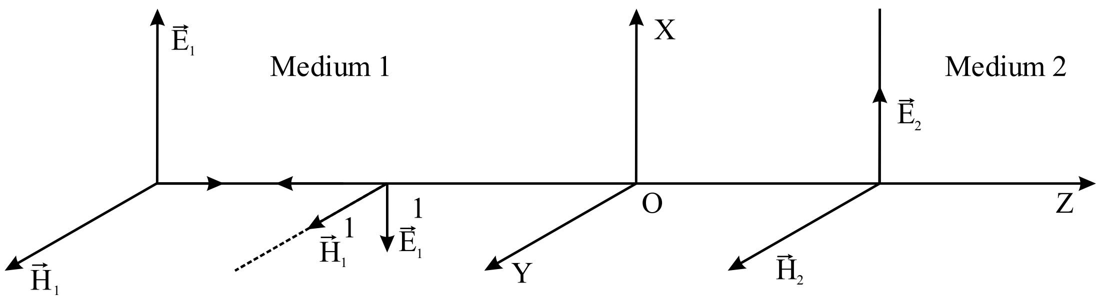


Fig. 6.8

The propagation constants are given by $k_1 = n_1 w/c$ and $k_2 = n_2 w/c$

From the boundary condition follows that tangential component of electric field is continuous at z = 0. so we have

$$E_{1x} - E'_{1x} = E_{2x} \quad \text{..........(6.12.1)}$$

We have from the boundary condition that tangential component of magnetic field is continuous at z = 0, then

$$H_{1y} - H'_{1y} = H_{2y}$$

or $\quad n_1 (E_{1x} + E'_{1x}) = n_2 E_{2x}$ ....(6.12.2)

Solving equations (6.12.1) and (6.12.2) we get,

$$E'_{1x} = \frac{n_2 - n_1}{n_2 + n_1}E_{1x} \quad \text{..........(6.12.3)}$$

$$E_{2x} = \frac{2\,n_1}{n_2 + n_1} E_{1x} \quad \text{.......(6.12.4)}$$

Fresnel co-efficient is defined as the ratio of reflected and wave and incident wave and ratio of transmitted wave and incident waves. These co-efficients are denoted by r and t.

We get,

$$r = \frac{E'_{1x}}{E_{1x}} = \frac{n_2 - n_1}{n_2 + n_1} \quad \text{.......(6.12.5)}$$

$$t = \frac{E_{2x}}{E_{1x}} = \frac{2\,n_1}{n_2 + n_1} \quad \text{.......(6.12.6)}$$

Now, the reflectance (or reflecting power) is defined as

$$R = \frac{\langle S'_1 \rangle}{\langle S_1 \rangle} \quad \text{.......(6.12.7)}$$

and transmittance as

$$T = \frac{\langle S_2 \rangle}{\langle S_1 \rangle} \quad \text{.......(6.12.8)}$$

where $\langle S_1 \rangle$, $\langle S'_1 \rangle$ and $\langle S_2 \rangle$ are the average energy. flows per unit area per unit time for the reflected and transmitted waves. So they represent the time averaged Poynting vectors or intensities of three waves. Also we know that $\langle S \rangle = \sqrt{\frac{\varepsilon}{\mu_0}} \frac{E_0^{\,2}}{2}$

$= \frac{n}{\mu_0 c} \frac{E_0^{\,2}}{2}$ where, $n = \sqrt{\frac{\varepsilon}{\epsilon_0}}$ is refractive index and E_0 is the amplitude of electric fields. So reflectance and transmittance in normal reflection is given by

$$R_n = \frac{E'^{\,2}_{1x}}{E^{\,2}_{1x}} = r^2 = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \quad \text{.......(6.12.9)}$$

and $$T_n = \frac{n_2 E_{2n}^{\,2}}{n_1 E_{1n}^{\,2}} = \frac{n_2}{n_1} t^2 = \frac{4 n_1 n_2}{(n_2 + n_1)^2} \quad \text{.......(6.12.10)}$$

From equations (6.12.9) and (6.12.10) we get $R_n + T_n = 1$....................(6.12.11)
which proves that energy is conserved at the interface.

6.13 Reflection and refractron at oblique incidence at the Interface Between two Dietectrics

The reflection and refraction of EM waves at a plane surface between two media of different dielectric properties are well known phenomena. The different aspects of the phenomena divide themselves into two classes.

1) Kinematatic Properties :

a) Angle of reflection equals angle of incidence.

b) Sneall's law : $\frac{\sin i}{\sin r} = \frac{n_2}{n_1}$ where i and r are the angle of incidence and reflection, while n_2 and n_1 are the corresponding indices of refraction.

2) Dynamic properties :

a) Intensities of reflected and refracted radiation.

b) Phase changes and polarization.

3) Polarization : Consider the incident, the reflected and the transmitted waves as shown in the Fig. 6.9. Here K_I, K_T, and K_R be the propagation vecitors for the incident, reflected and transmitted waves, respectively. From boundary conditions, we know that all rays must have the same angular frequency. The electric and the magnetic field vectors can be written as follows :

For parallel polarization (P)

Incident wave :

$$\left.\begin{aligned} \vec{E}_I(\vec{r}.\ t) &= \vec{E}_{0I} e^{\,i(\vec{k}_I.\vec{r}-wt)} \\ \vec{H}_I(\vec{r}.\ t) &= \left(\frac{n_1}{\mu_0 c}\right)\hat{k}_1\ x[E_{0I} e^{\,j(\vec{k}_I.\vec{r}-wt)}] \end{aligned}\right\} \quad \text{......................(6.13.1)}$$

Refected wave :

$$\left.\begin{aligned} \vec{E}_R(\vec{r}.\ t) &= \vec{E}_{0R} e^{\,j(\vec{k}_R.\vec{r}-wt)} \\ \vec{H}_R(\vec{r}.\ t) &= \left(\frac{n_1}{\mu_0 c}\right)\hat{k}_1'\ x[E_{0R} e^{\,j(\vec{k}_R.\vec{r}-wt)}] \end{aligned}\right\} \quad \text{......................(6.13.2)}$$

Transmitted wave :

$$\left.\begin{array}{l}\vec{E}_T(\vec{r}.\ t) = \vec{E}_{0T} e^{j(\vec{k}_T.\vec{r}-wt)} \\ \vec{H}_T(\vec{r}.\ t) = \left(\frac{n_2}{\mu_0 c}\right) \hat{k}'_2 x[E_{T0} e^{j(\vec{k}_T.\vec{r}-wt)}]\end{array}\right\} \text{......................(6.13.3)}$$

Where $\hat{k}_1, \hat{k}'_1, \hat{k}_2$ are the unit vectors along $\vec{K}_I$, $\vec{K}_R$ and $\vec{K}_T$ respectively let θ_I, θ_R and θ_T be the angles between the normal to the interface and the propagation direction. The angles θ_I, θ_R and θ_T are called angle of incidence, reflection and refraction, respectively.

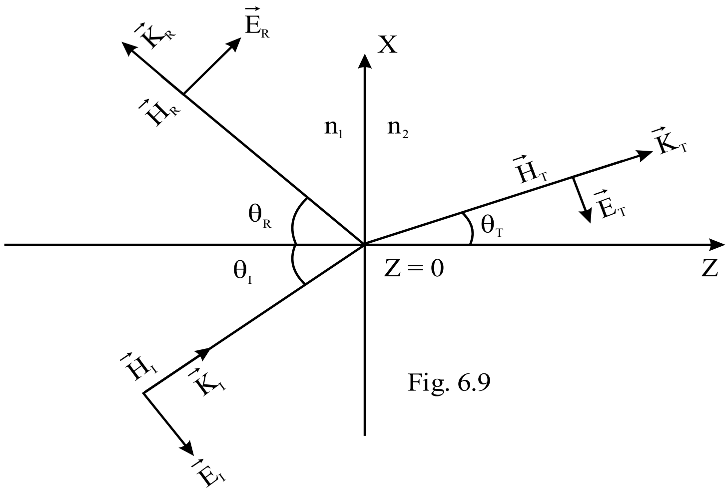


Fig. 6.9

Reflection and Retraction for oblique incidence: The plane of incidence XZ plane, The incident Electric field as in the xz plane (P polarization).

All three waves have the same frequency that is determined once and for the source then three wave numbers are related by

$$K_I\,\upsilon_I = K_R\,\upsilon_I = K_T\upsilon_2$$

$$\text{or} \qquad K_I = K_R = \frac{\upsilon_2}{\upsilon_1} K_T = \frac{n_1}{n_2} K_T \text{...(6.13.4)}$$

The existence of boundary conditions at Z = 0, which must be satisfield at all points at all times, implies that the spatial (and time) variation of all the fields must be the same at z = 0, consequently, we must have the phase factors all equal at z = 0. For the spatial terms, evidently

$$\vec{K}_I.\vec{r} = \vec{K}_R.\vec{r} = \vec{K}_T.\vec{r} \quad \text{when } z = 0$$

which is known as phase matching ....(6.13.5)

Explicitily,

$$x\,(K_I)_x + y\,(K_I)_y = x\,(K_R)_x + y\,(K_R)_y = x\,(K_T)_x + y\,(K_T)_y \quad \text{..................(6.13.6)}$$

for all x and all y

Equation (6.13.6) can only hold if the components are separately equal, for if $x = 0$, we get

$$(K_I)_y = (K_R)_y = (K_T)_y \quad \text{..........(6.13.7)}$$

while $y = 0$, gives

$$(K_I)_x = (K_R)_x = (K_T)_x \quad \text{..........(6.13.8)}$$

let us orient all our axes so that $\vec{K}_I$ lies in the xz plane [ie.$(K_I)_y = 0$]; so from equation (6.13.7) it follows that $\vec{K}_R$ and $\vec{K}_T$ also lies in the same plane. Thus we conclude that

First law : The incident, reflected and transmitted wave vectors form a plane

From equation (6.13.8) it follows that

$$K_I \sin\theta_I = K_R \sin\theta_R = K_T \sin\theta_T \quad \text{..........(6.13.9)}$$

Second law : The angle of incidence is equal to the angle of reflection

$$\theta_I = \theta_R \quad \text{..........(6.13.10)}$$

which is the law of reflection

Now for the transmitted angle

Third law,

$$\frac{\text{Sin}\theta_T}{\text{Sin}\theta_I} = \frac{n_1}{n_2} \quad \text{..........(6.13.11)}$$

which is the snell's law of refraction. The boundary condition that the tangential component of the electric field is continuous across the interface ($z = 0$) gives

$$E_{0I}\cos\theta_I - E_{0R}\cos\theta_R = E_{0T}\cos\theta_T \quad \text{..........(6.13.12)}$$

The boundary condition that the tangential component of the magnetic intensity is continuous across the interface ($z = 0$) gives

$$n_1 E_{0I} + n_1 E_{0R} = n_2 E_{0T}$$

$$n_1 (E_{0I} + E_{0R}) = n_2 E_{0T} \quad \text{..........(6.13.13)}$$

Equations (6.13.12) and (6.13.13) can be solved for $\frac{E_{0R}}{E_{0I}}$ and $\frac{E_{0T}}{E_{0I}}$ which gives the Fresnel's reflection and transmition co-efficients

$$r_p = \frac{(n_2 \cos\theta_I - n_1 \cos\theta_T)}{(n_2 \cos\theta_I + n_1 \cos\theta_T)} \qquad \text{.......(6.13.14)}$$

$$t_p = \frac{2n_1 \cos\theta_I}{(n_2 \cos\theta_I + n_1 \cos\theta_T)} \qquad \text{.......(6.13.15)}$$

Eliminating the refracting index from the above equations (6.13.14) and (6.13.15) we get

$$r_p = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)} \qquad \text{.......(6.13.16)}$$

$$t_p = \frac{2\cos\theta_I \ \mathrm{Sin}\theta_T}{\mathrm{Sin}(\theta_I + \theta_T)\ \mathrm{Cos}(\theta_I - \theta_T)} \qquad \text{.......(6.13.17)}$$

Note : r_p given by equation (6.13.16) is drawn r_p verses θ_I [Fig. 6.12 (a)]

S Polarization: for S polarization electric field of the incident wave is nornal to the plane of incidence consequently, magnetic field in the plane of incidence. All the reflected and transmitted waves are shown in fig. 6.10. Frequencies of the three waves are the same, Like P wave, the boundary conditions remains the same at z = 0, the spatial and time variation of all fields must be the same. Consequentily phase factor are all equal at z = 0 i.e.

$$(K_I.\vec{r})_{z=0} = (K_R.\vec{r})_{z=0} = (K_T.\vec{r})_{z=0} \qquad \text{.......(6.13.18)}$$

Thus the law of reflection ($\theta_I = \theta_R$) and Snell's law ($n_1 \mathrm{Sin}\theta_I = n_2 \mathrm{Sin}\theta_T$) are obtained for S- polarization. Also phase matching confirms that all the wave vectors $\vec{K}_I$, $\vec{K}_R$ and $\vec{K}_T$ are coplanar.

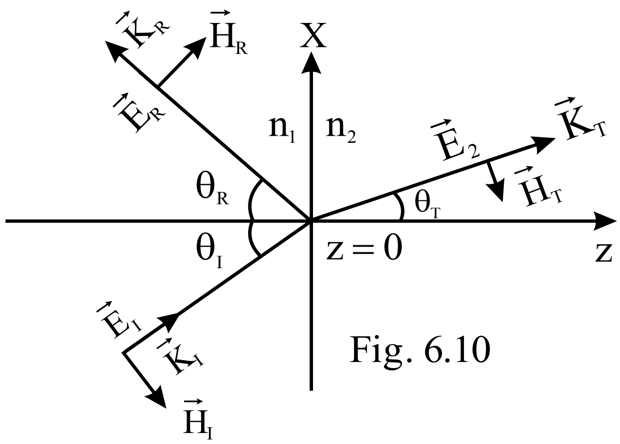


Fig. 6.10

* Reflection and refraction for oblique incidence, the plane of incidence (plane of the paper xz). The incident electric field is perpendicular

to the x z plane (S polarization)

The continuity of the tangential components of $\vec{E}$ and $\vec{H}$ gives (at z = 0)

$$E_{0I}e^{j(\vec{K}_I \cdot \vec{r})} + E_{0R}e^{j(\vec{K}_R \cdot \vec{r})} = E_{0T}e^{j(\vec{K}_T \cdot \vec{r})}$$

$$\Rightarrow E_{0I} + E_{0R} = E_{0T} \quad \text{......(6.13.19)}$$

and $n_1E_{0I}\cos\theta_I\, e^{j(\vec{K}_I \cdot \vec{r})} - n_1E_{0R}\cos\theta_R\, e^{j(K_R \cdot \vec{r})} = n_2E_{0T}\cos\theta_T\, e^{j(K_T \cdot \vec{r})}$

$$\Rightarrow n_1E_{0I}\cos\theta_I - n_1E_{0R}\cos\theta_I\, E_{0R} = n_2E_{0T}\cos\theta_T \quad \text{......(6.13.20)}$$

Solving (6.13.19) and (6.13.20) for $\frac{E_{0R}}{E_{0I}}$ and $\frac{E_{0T}}{E_{0I}}$ We get the Fresnel's reflection and transmission coefficients.

$$r_s = \frac{E_{0R}}{E_{0I}} = \frac{n_1\cos\theta_I - n_2\cos\theta_R}{n_1\cos\theta_I + n_2\cos\theta_R} \quad \text{......(6.13.21)}$$

and $$t_s = \frac{E_{0T}}{E_{0I}} = \frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T} \quad \text{......(6.13.22)}$$

Again utilising Snell's law, equation (6.13.21) and (6.13.22) can be written as

$$r_s = \frac{\mathrm{Sin}(\theta_I - \theta_T)}{\mathrm{Sin}(\theta_I + \theta_T)} \quad \text{......(6.13.23)}$$

$$t_s = \frac{2\cos\theta_I\,\mathrm{Sin}\theta_T}{\mathrm{Sin}(\theta_T + \theta_T)} \quad \text{......(6.13.24)}$$

Note r_s given by equation (6.13.23) is drawn r_s verses θ_I [Fig. 6.12 (a)]

Reflectance is the amount of flux (radiation) reflected by a surface, normalised by the amount of flux incident on it. Transmittance is the amount of flux (radiation) transmitted by a surface, normalised by the amount of flux incient on it. So if $<\vec{S}_{IS}>$ is the time averaged Poynting vector for the incident wave for s polarization and $<\vec{S}_{RS}>$ for the reflected wave, reflectance is given by

$$R_S = \left| \frac{\hat{n} <\vec{S}_{RS}>}{\hat{n} <\vec{S}_{IS}>} \right| \quad \text{...(6.13.25)}$$

If $<\vec{S}_{TS}>$ is time - averaged Poynting vector for the transmitted wave, then transmittance

$$T_S = \left| \frac{\hat{n} <\vec{S}_{TS}>}{\hat{n} <\vec{S}_{IS}>} \right| \quad \text{...(6.13.26)}$$

Similarly, the reflectance and the transmittance for P polarization are as follows-

$$R_P = \left| \frac{\hat{n}.<\vec{S}_{RP}>}{\hat{n}.<\vec{S}_{IP}>} \right| \quad \text{...(6.13.27)}$$

$$T_P = \left| \frac{\hat{n}.<\vec{S}_{TP}>}{\hat{n}.<\vec{S}_{IP}>} \right| \quad \text{...(6.13.28)}$$

So the Fresnel's coefficients are

$$R_s = r_S^{\,2} = \left[\frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T}\right] \quad \text{...(6.13.29)}$$

$$T_s = \frac{n_2\cos\theta_T}{n_1\cos\theta_I}\ t_S^{\,2} = \frac{4n_1 n_2\cos\theta_I\cos\theta_T}{(n_1\cos\theta_I + n_2\cos\theta_T)} \quad \text{......................................(6.13.30)}$$

$$R_P = <r_P>^2 = \left(\frac{n_2\cos\theta_I - n_1\cos\theta_T}{n_2\cos\theta_I + n_1\cos\theta_T}\right) \quad \text{...(6.13.32)}$$

$$T_P = \frac{n_2\cos\theta_T}{n_1\cos\theta_I}\ t_P^{\,2} = \frac{4n_1 n_2\cos\theta_I\cos\theta_T}{(n_2\cos\theta_I + n_1\cos\theta_T)^2} \quad \text{....................................(6.13.33)}$$

Note : See Fig. 6.12 (b) for R_S, R_P, T_S, T_P plotted against angle of incidence.

Key points to be take away :

1. Both the coefficients (R & T) are only independant of the material properties i.e permeability (as per second form the equations), throgh still have same implications of the reflective index.

2. Both the coefficients (R & T) are only dependent on the angle of incident θ_I and angle of refraction θ_R

Some interesting results to observe :

1. Normal incidence : Here $\theta_I = \theta_T$ so from equations (6.13.14), (6.13.15), (6.13.21) we get,

$$r_P = -r_S = \frac{n_2 - n_1}{n_2 + n_1} \qquad (6.13.33)$$

and $$t_P = t_S = \frac{2n_1}{n_2 + n_1} \qquad (6.13.34)$$

Also $R_S = R_P$ and $T_S = T_P$

2. Grazing angle of incidence ; In this case incident waves touches the interface at on angle $\theta_I \approx \pi/2$, then

$$r_P = - r_S = -1 \qquad (6.13.35)$$

and $$t_P = t_S = 0 \qquad (6.13.36)$$

so that $$R_P = R_S = 1 \qquad (6.13.37)$$

$$T_P = T_S = 0 \qquad (6.13.38)$$

This reaveals that there is total reflection for both S and P polarization. Just think of a beam of light shinning on a flat surface

3. Brewoters law : A reletionship for light waves stating that the maximum polarization (vibration in one plane only) of a ray of light may be acheived by letting the ray incident on a surface of transpent medium in such away that the refracted ray makes

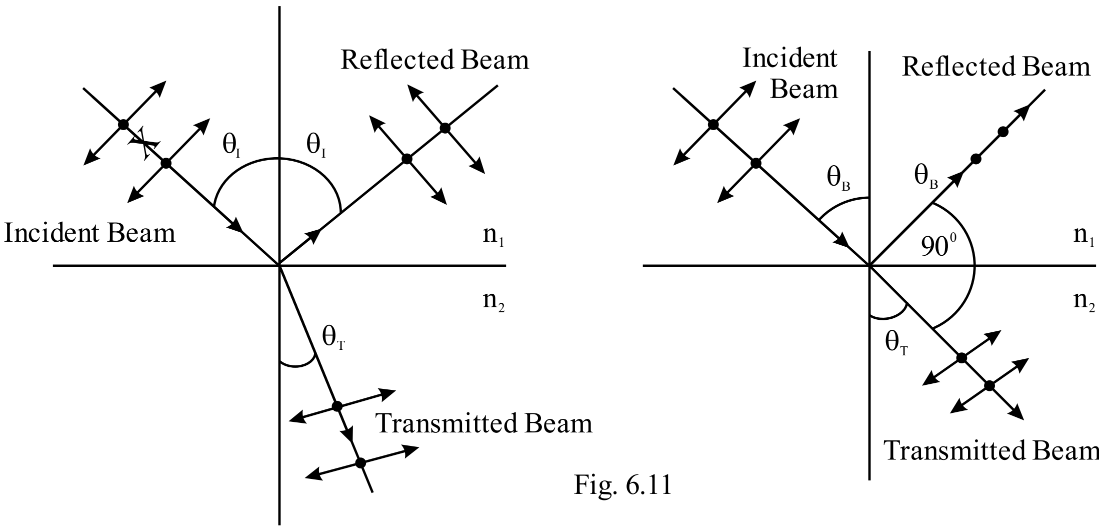
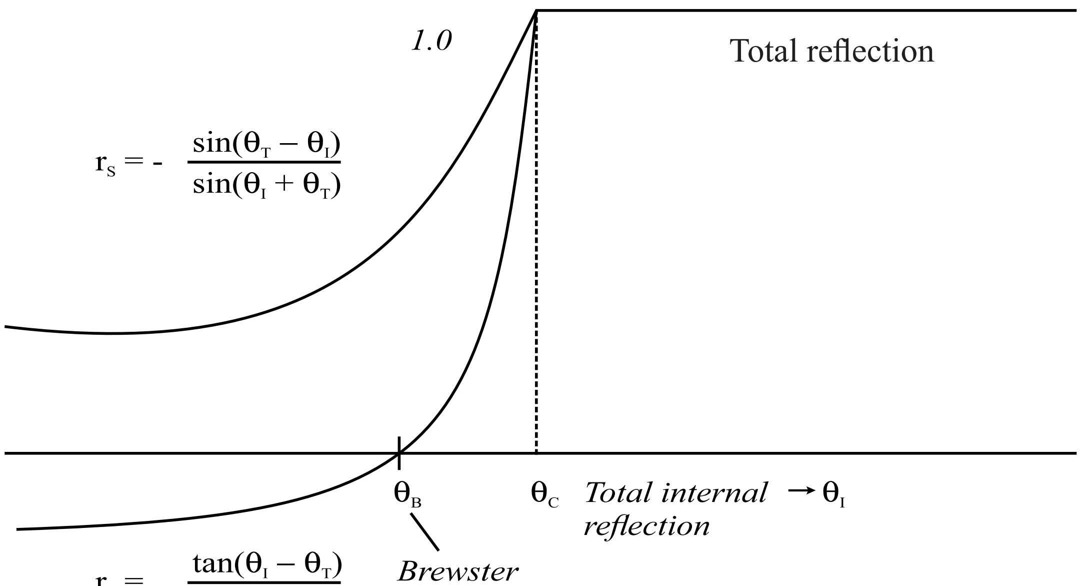
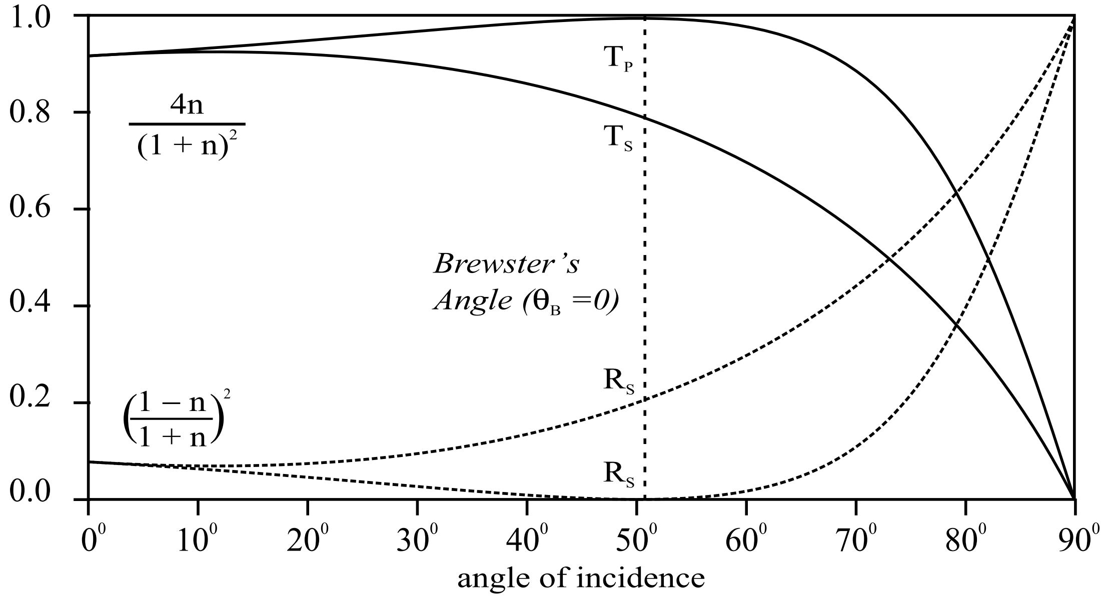


Fig. 6.11



(a)



(b)

fig. 6.12

on angle of 90^0 with the reflected ray.

From equation (6.13.16), we find that for $(\theta_I + \theta_T) = \pi/2$ $r_P = 0$ which implies that for $(\theta_I$

$+ \theta_T) = \pi/2$, the electric field polarized parallel to the plane of incidence is not reflected at all. Under this condition reflection coefficients $r_S \neq 0$ ie the electric field polarized normal to the plane of incident is partly reflected.Thus an unpolarized light consiting of both types of vibration of E fields incident at angle. θ_B satisfying the condition $(\theta_I + \theta_T) = \pi/2$, will be plane polarized normal to the plane of incidence. This angle of incidence θ_B for which $r_P = 0$ is known as Brewsteis angle under this condition, from Snell's law, we have

$$\frac{n_2}{n_1} = \frac{\text{Sin}\theta_I}{\text{Sin}\theta_T} = \frac{\text{Sin}\theta_B}{\text{Sin}(\pi/2 - \theta_B)} = \tan\theta_B \quad \text{...(6.13.39)}$$

Example of Breswter law application is polarized sunglasses, These glasses use the principle of Brewster angle. The polarized glasses reduce glare that is directly from the sun and also from the horizental surface like road and water.

Total Internal Reflection, Evanescent Wave :

According to Snell's law, $\frac{\text{Sin}\theta_I}{\text{Sin}\theta_T} = \frac{n_2}{n_1}$

So when light wave passes from a optically denser medium into a rarer one ie. $n_1 > n_2$ the wave vector $\vec{K}$ bends away from the normal. Specifically, if the light is incident at the ceitical angle θ_C defined as $\theta_C = \frac{\text{Sin}^{-1} n_2}{n_1}$ we get $\text{Sin}\theta_T = 1$, or $\theta_T = \frac{\overline{\wedge}}{2}$, which implies that, the trasmitted ray just grazes the surface. It $\theta_I > \theta_C$, then $\text{Sin}\,\theta_T > 1$, which implies that it dies not correspond to any possible θ_T. Here no rays are reflected, rather the whole light wave reflected back to the denser medium. This phenomenon is called total internal reflection.

In spite of no reflection into the denser medium, The fields are not zero in that medium, which is called evanescent-wave. It altenuates rapidly and it transports no energy into the rarer medium.

Transmitted wave vector can be written as

$$\vec{K}_T = K_T\,(\text{Sin}\theta_T\,\hat{i} + \cos\theta_T\,\hat{K}) \quad \text{..(6.13.40)}$$

$$\text{with} \quad K_T = \frac{\omega}{v_2} = \frac{\omega n_2}{C} \quad \text{..(6.13.41)}$$

As $\sin\theta_T > 1$, obviously we can write $\cos\theta_T = \sqrt{1-\text{Sin}^2\theta_T}$, which is maginary number.

Now, for the transmitted wave $\vec{E}_T = \vec{E}_{0T} c^{j(\vec{k}_T.\vec{r}-\omega t)}$

where $\vec{K}_T . \vec{r} = K_T \text{Sin}\theta_T x + K_T \cos\theta_T Z$

$$= \frac{\omega n_2}{C} \text{Sin}\theta_T\, x + j\, \frac{\omega n_2}{C} \sqrt{\text{Sin}^2\theta_{T-1}}\, Z$$

$$= \frac{\omega n_1}{C} \text{Sin}\theta_I\, x + j\, \frac{\omega}{C} \sqrt{(n_1^2 \text{Sin}^2\theta_I - n_2^2)}\, Z$$

$$= K^* x + j\, k\, z$$

Where $K^* = \dfrac{\omega n_1\, \text{Sin}\theta_I}{C}$ and $K = \dfrac{\omega}{C}\sqrt{n_I^2 \text{Sin}^2\theta_I - n_2^2}$

So, we can write the tranmittes wave as

$$\vec{E}_T(\vec{r}.\, t) = \vec{E}_{0T} e^{j(\vec{k}_T.\vec{r}-wt)}$$

or

$$E_T(\vec{r}.\, t) = \vec{E}_{0T} e^{-k_2}\, e^{i(k^* x-\omega t)} \quad \text{...(6.13.41a)}$$

This is the wave defined as evanescent wave propagating in x-radiation ie. parallel

successive internal reflections.

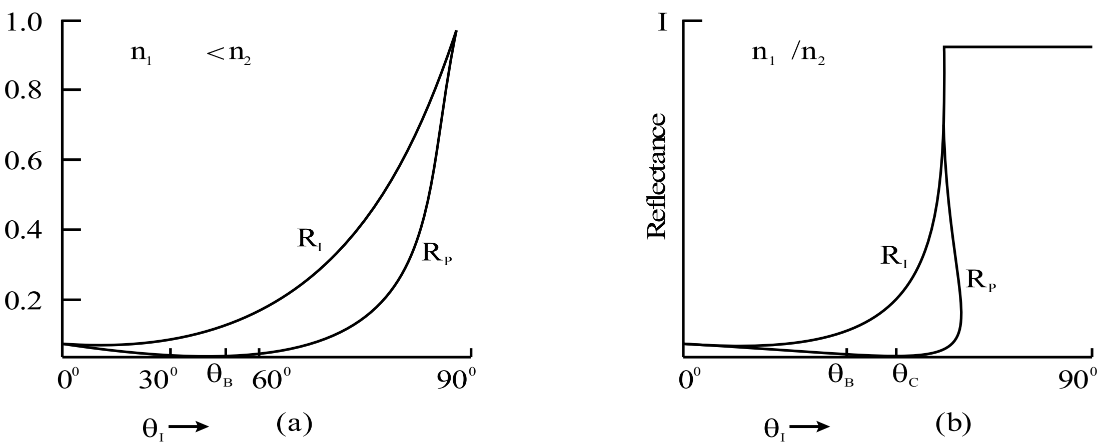


Fig. 6.13

to x direction with a penetration depth of K^{-1}. It decays rapidly and becomes negligible beyond a distance of few wavelengts.

Reflectances for S and P polarizations when $n_1<n_2$ and $n_1>n_2$

Let $\cos\theta_T = \sqrt{1-\mathrm{Sin}^2\theta_T} =$ jD. Now for parallel and perpendicular, electric field vectors, reflection coeffcient becomes from, equation (6.13.14) and (6.13.21)

$$r_p' = \frac{n_2\cos\theta_I - jn_1\Delta}{n_2\cos\theta_I + in_1\Delta} \quad \text{.......(6.13.41)}$$

and $$r_s' = \frac{n_1\cos\theta_I - jn_2\Delta}{n_1\cos\theta_I + jn_2\Delta} \quad \text{.......(6.13.42)}$$

Then Reflectance is given by,

$$R_p' = |r_p'|^2 \quad \text{.......(6.13.43)}$$

$$R_S' = |r_S'|^2 \quad \text{.......(6.13.44)}$$

It folloes fro above equations that

$$R_p' = R_S' = 1 \quad \text{.......(6.13.45)}$$

From Fig. 6.13 (a) it is clear that the reflectance for S and P polarizations when $n_1<n_2$; it shows that there is no total internal reflection. From Fig. 6.13 (b) it is clear that when $n_1>n_2$: there is a critical angle qc and total internal reflection.

From, equations (6.13.41) and (6.13.42), it can be written in phase from,

$$r_P' = e^{(-j2\phi)} \quad \text{.......(6.13.46)}$$

and $$r_S' = e^{(-j2\phi')} \quad \text{.......(6.13.47)}$$

where $\tan\phi = n_1\Delta/n_2\cos\theta_1$ and $\tan\varphi' = \dfrac{n_2\Delta}{n_1\cos\theta_I}$. Here, the electric field lags that incident wave by 2ϕ, for P polarization and $2\phi'$ for S-polarization respectively. elearlly elliptically polarized light will be observed it the incident wave is polarized is a palne making on angle $(\neq 90)^0$ with the plane of incidence.

6.14 Summary

1. We have studied that how Maxwell's equation and its solution proved the secmingly disparate phenomen of electricity magnetism, and optics are all related aspect

of the larger phenomenon of electro magnetics. Solutions to the fundamental euations of electrialy and magnetism are electromagnetic waves. Most importants findings of the solution os Maxwell's equations is the revelation that all forms of electromagnetic wave be it, visible light x-rays, r-ray, infrared on ultraviolet light, propagle at the speed of light in vaccum, and transference of energy from one space to another without any medium.

2. We have also seen, how Maxwell's moclified the Ampere's law of steady flow current to the case of varying current with the introduction of defination of displacement current due to changing electric field. Instantaneous magnetic field generated due to changing electric field has led to the propagation of radiation, which carries energy from one place to another. The EM radiation which propagaties energy is a major discovery by Maxwell's due to his prediction of displacement current Modern age comminacation is impossible with out EM radiation.

3. We have also studied the introduction of gauge transformation. We have discussed the six variable of EM fields can be represented.

4. The tangential componet of the electric field ($\vec{E}$) is continuous across the interface.When the medium conductivity infinity, the tangential component of magnetic intennity ($\vec{H}$) is continuous accross the interface

5. We have discussed reflection and refraction at the plane interface of two non conducting (di electric) media (i) Normal inciedence and (ii) oblique incidence. We have calculated the Fresnel reflection coefficient and Fresnel transmission cofficient for both normnal and oblique incidence . We have also evaluated reflectances for s and p polarizations when (a) $n_1<n_2$ (when there is no total internal reflection) and (b) $n_1>n_2$ (when there is critical angle θc and total internal reflection).

6. We have derived Brewster's law $\tan\theta_B = \tan^{1}\frac{n_2}{n_1}$

6.15 Review Questions and Answer

Question :

1. S now that the dispalcement current in a parallel plate capacitor is equal to the conduction current in the connecting leads.

Answer :

The capcitance of a parallal plate capacitor is $C = \varepsilon\frac{A}{d}$

where A is the area of plate d is the distance between them, and ε is its permittivity. The conducting current in the conecting leads is

$$i_e = \frac{dq}{dt} = C\,\frac{dv}{dt}$$

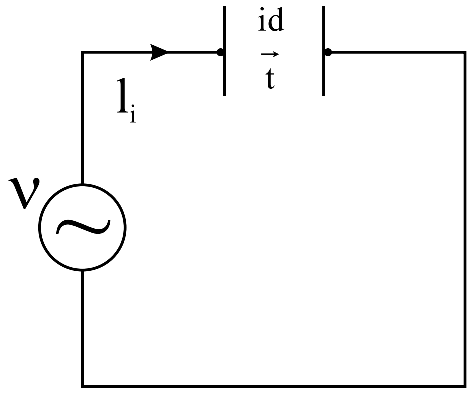
or $$i_e = \varepsilon\frac{A}{d}\,\frac{dv}{dt}$$

The electric field in the (capator) dielectric E = n/d. Now electric displacement is

$$D = \varepsilon E = \varepsilon\frac{\nu}{d}$$ *, Hence the displacement current density* $$\frac{dD}{dt} = \frac{\varepsilon}{d}\,\frac{dv}{dt}$$

The displacement current is

$$i_d = A\frac{dD}{dt} = \frac{A}{d}\,\varepsilon\,\frac{dv}{dt}$$



Hence $i_D = i_C$

2. State Poynting vector

Answer : See article 6.2

3. Define Brewster's Law

Answer : See article 6.13

4. Define critical angle penetrating depth.

Answer : See section 6.13

5. Define, momentum, pressure and angular momentum of electromagnetic radiation.

Answer : Radiatopn pressure is the mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field. This includes the momentum of EM radiation of any wave lenght is obsoerd reflected or emitted by matter on any scale.

For further follow-up answer, see article 6.5

6. State the boundary conditions between two interfacing different dielectric.

Answer : See article 6.8

7. State the boundary conditions between dielectric and conductind media

Answer : See article 6.8

8. Prove that the momentum density stored in an electromagnetic field is given $\vec{g} = \vec{s}/c^2$ in free space where $\vec{S} = \vec{E} \times \vec{H}$ Poynting vector.

Answer : Force $\vec{F} = \dfrac{d(P_{mechanical})}{dt}$ from this we can show that

$$\vec{P}_{\text{electromagnetic}} = \vec{P}_{em}$$

$= \mu_0 \varepsilon_0 \int s d\nu$ = momentum related to electromagnetic wave

$\therefore$ momentum density $\vec{P}em = \mu_0 \varepsilon_0 \vec{S}$

Again $\dfrac{1}{\mu_0 \epsilon_0} = \dfrac{1}{c}$

So, momentum density $\vec{P}_{em} = \dfrac{\vec{S}}{c^2} = \vec{g}$

6.16 Problems and Solutions

A Steady current I is flowing through a metallic wire of length L and radius R throght a potential difference V calculate (a) Poynting vector, (b) Total energy delivered to the system and, (c) derive the value of the resistance of the wire R using $\int \vec{J}, \vec{E}, d\nu$.

Solution :

Assuming electric field E is parallel to the wire then, $E = \dfrac{V}{L}$ the magnetic field is circumtanential at the surface

$$B = \frac{\mu_0 I}{2\wedge R}$$

Hence the Poynting vector magnetude is

$$S_0 = \frac{I}{\mu_0} \left| \vec{E} \times \vec{B} \right|$$

(a) $$= \frac{I}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\wedge R} = \frac{VI}{2\wedge RL}$$

and it shows that Poynting vector is inward

(b) The energy passing throngh the surface of the wire

$$\int \vec{S}_0 . \vec{ds} = - S_0 2\pi RL$$
$$= \frac{-VI}{2\pi RL} 2\pi RL$$
$$= -VI$$

(c) Now $\int (\vec{J}.\vec{E})\, dv = \int \sigma . E^2 dv = \sigma \frac{V^2}{L^2} \int dv$

$$= \sigma \frac{V^2}{L^2} 2\pi R^2 L$$
$$= \sigma \frac{V^2}{L} \pi R^2 =$$
$$\therefore \int (\vec{J}.\vec{E})\, dv = \frac{\pi R^2 V^2 \sigma}{L}$$

As $\int (\vec{J}.\vec{E})\, dv$ = total Joule loss per unit time due to the flow os current

$$\int (\vec{J}.\vec{E})\, dv = \frac{V^2}{R}$$
$$\therefore R = \frac{L}{\pi R^2 \sigma}$$

2. A plane electromagnetic wave has the magnetic field given by

$$\vec{B}(x,y,z,t) = B_0 \operatorname{Sin}\left[\left(\frac{(x+y)K}{V^2} + \omega t\right]\hat{K}$$

Where K is the ware number, where $\hat{i}, \hat{j}, \hat{k}$ are the cartensian unit vectors in x, y and z directions respectively.

(a) Find the electric field $\vec{E}$ (x,y,z,t)

(b) Find the average Poynting vector

Solution :

(a) $\vec{E} = -\frac{c}{k}(\vec{K} \times \vec{B})$

$$= -\frac{c}{k}\left[K\frac{(i+j)}{V^2} \times B_0 \, Sin\left\{(x+y)k + \omega t\right\}\hat{K}\right]$$

$$\vec{E} = C\,B_0\left[\frac{(x+y)}{V^2}k + wt\right]\left(\frac{\hat{i} - \hat{j}}{V^2}\right)$$

(b) The average Poynting vector is given by

$$\bar{S} = \frac{C\,B_0^2}{2\mu_0}\hat{K} = \frac{C\,B_0^2}{2\mu_0} x - \left(\frac{\hat{i}+\hat{j}}{V^2}\right)$$

$$= \frac{C\,B_0^2}{2\mu_0} x \left[\frac{\hat{i}+\hat{j}}{V^2}\right]$$

3. The space time dependence of the electric of a linearly polarzed light in free space is given by $\hat{i}\,E_0$ Cos (ωt-kz). Find the time average density associate with electric field.

Solution :

$$u_E = \tfrac{1}{2}\, e_0 E_2 = \tfrac{1}{2}\, e_0 E^2 \cos^2(\omega t - kz)$$

$$\therefore \langle u'_E \rangle = \tfrac{1}{4}\, e_0 E_0^{\,2}$$

4. A plane polarized electromagnetic wave in free space at time t = 0, is given by $\vec{E}$ (x,z,t) = j exp[j(6x+8z)]

Solution : Magnetic field vector is given by

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$$

$$= \frac{1}{c}\left(\frac{6\hat{i} + 8\hat{k}}{10}\right) \times 5\hat{j}\ \exp[j(\vec{k},\vec{r} - \omega t]$$

$$= \frac{2}{c}\left(6\hat{k} - 8\hat{i}\right)\exp[j(6x+8z)-10ct]$$

5. It the vector potential $\vec{A} = \beta x\ \hat{i} + 4y\hat{j} - 5z\,\hat{k}$ satisfied the coulomb gauge find the value of the constant.

Solution : condition for Coilomb gauge is $\vec{\nabla}.\vec{A} = 0$

$$\Rightarrow \beta + 4 - 5 = 0$$
$$\Rightarrow \beta = 1$$

6. A vector potential $\vec{A} = ke^{-at}\, r\, \hat{r}$ (where a and k are constants) corresponding to an electromagnetis field is changed to $\vec{A} = -\, ke^{-at}\, r\, \hat{r}$. Prove that. This will be a gauge transformation if the corresponding change φ' - φ in the scalar potential is - akr^2e^{-at}

Solution : Gauge transformation

$$\vec{A} = \vec{A} + \vec{\nabla}\,\lambda\,,\ \varphi' = \varphi - \frac{\partial\lambda}{\partial t}$$
$$\Rightarrow A' - A = -\,2ke^{-at}\, r\,\hat{r} = \vec{\Delta}\,\lambda = \frac{\partial\lambda}{\partial r}\,\hat{r}$$
$$\Rightarrow \lambda = -ke^{-at}\, r^2 \Rightarrow \frac{\partial\lambda}{\partial t} = kae^{-at}\ r^2$$
$$\Rightarrow \phi' - \phi = -\,\frac{\partial\lambda}{\partial t} = -\,kae^{-at}r^2$$

7. The intensity of sunlight reaching the earthis surface is about 1300 wm-2. Calculate strenght of the electric and magnetic fields of the incoming sunlight.

Solution : The time average Poynting vector

$$<\vec{S}> = \tfrac{1}{2}\, Re(\vec{E} \times \vec{H})$$

Taking incoming wave variation $\vec{E} = E_0e^{-jwt}$ and $\vec{H} = H_0e^{-jwt}$.

we get $S_{av} = |<\vec{S}>| = \frac{1}{V_2}\, E_0 \times \frac{H_0}{\sqrt{2}} = E_{rms}.\, H_{rms}, B = \mu_0 H = \frac{E}{C}$

or, $H_{rms} = \frac{E_{rms}}{\mu_{oc}}$

So, $S_{av} = \frac{E_{rms}{}^2}{c\mu_0}$

on $E_{rms} = \sqrt{c\mu\ S_{av}} = \sqrt{3\times10^{8}\times4\pi10^{-7}1300}\ \ Vm^{-1}$

$=7000Vm^{-1}$

Again $B_{rms} = \frac{E_{rms}}{c} = \frac{700}{3 \times 10^8}\, T$

$= 2.33 \times 10^{-6}\ T$

Unit 7 ❑ Network Theorems

Structure

7.1 Objectives

You will know from this unit—

- To learn teachniques of solving circuits for bilinar network comprising passive elements,
- Application of KVL-KCL, in series, parallel, voltage and current divider rule, source transformation techniques,
- Study of Thevenin, Norton, Superposition, Reciprocity Theorem and Power Transfer Theorem, and their equivalent circuits, to simply the evaluation process,
- Necessity of Thevenin's and Norton's Theorem in A.C. circuit behaviour and analysis.

7.2 Introduction

Network theorems give a more simple way to analyse electrical circuits than Ohm's law or Kirchhoff's laws. They are not basic theorems and are deducible from Kirchhoff's laws.

To begin with, the details we focus on some relevant definitions.

Electric network :

Electric network is combination of electric elements like cells, resistances, capacitors, inductances, diodes, transistors etc.

An **active network** is that which carries source (sources) of emf, like cell, transistor etc.

A **passive network work** does not carry **active elements**—the source of emfs.

Electric circuit :

Electric circuit is a closed path through which electric current flows or intended to flow. A closed circuit is that through current flows and when no current flows through, it is an open circuit. Two points are said short circuit when zero impedance joins the two points. A *linear* circuit is that where in the circuit elements do not change with voltage or current, otherwise it is a non-linear circuit. Node is point at junction of two or more circuit elements.

The Sources:

Voltage Source

A voltage source is an emf generator. The figure (7.1) shows an emf generator with emf E, internal resistance r_i and load resistance R_L. i be the current flowing through the circuit. Then $E = i(r_i+R_L) = ir_i + V_0$, so the output voltage is less than the input due to internal potential drop $V_i = ir_i$. To make the output voltage independent of current r_i should tend to zero. A voltage source with zero internal resistance is an ideal voltage source.

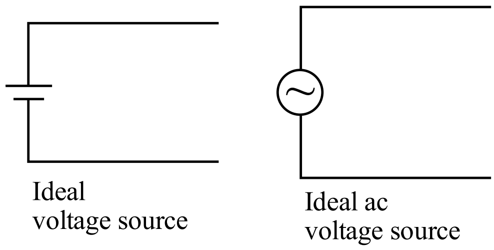


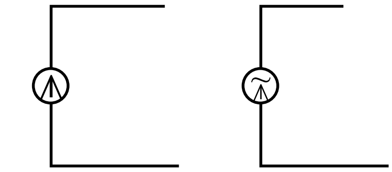
Fig 7.1

Current Source :

A current source is a current generator. Reference to fig (7.2), we can write

$$i = \frac{E}{r_i + R_L} = \frac{E}{r_i(1+\frac{R_L}{r_i})}$$

Fig 7.2



So the source current become independent of load resistance as r_i tends to infinity.

An ideal current source has infinite internal resistance.

Thevenin's theorem

The theorem states that any two terminal linear, bilateral network with impedances and energy sources can be replaced by an open circuit voltage V_{Th} (called Thevenin's voltage) generator across the terminals with an internal impedance Z_{Th} (Thevenin's impedance) measured across the terminals replacing the energy sources by their respective internal impedances. Then the current through the load resistance Z_L will be $i_i = \frac{V_{Th}}{Z_{th} + Z_i}$

Fig. (7.3) illustrate the procedure of Thevenin's theorem.

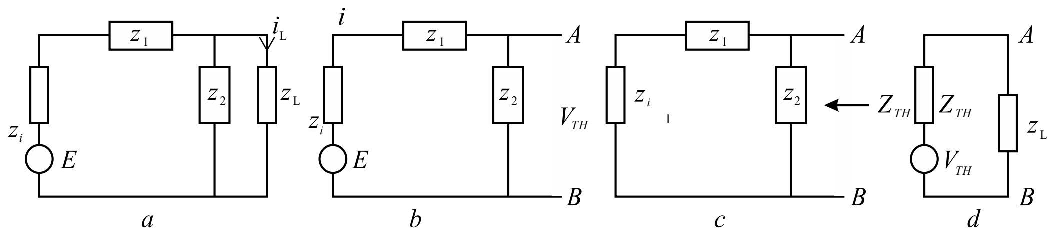
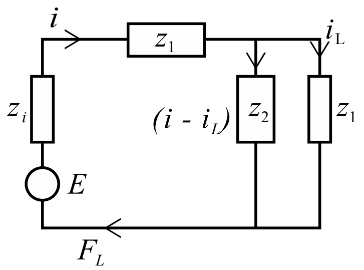


Fig. 7.3

a) The circuit for which i_L is to be determined, z_i is the internal impedance of the voltage source E.

b) Load z_L is removed and the Thevenin's voltage $V_{Th} = \frac{Ez_2}{z_i + z_1 + z_2}$ is calculated about AB.

c) The E is short-circuited and $z_{Th} = \frac{z_2(z_i + z_1)}{z_i + z_1 + z_2}$ is calculated across the terminal AB.



d) The Thevenin's equivalent circuit.

Proof of Thevenin's theorem (using Kirchhoff's laws)

The adjoining figure is a replica of Fig.(7.4) with the currents flowing through the Given impedances). Using KVL and KCL in the circuit, we have,

Fig 7.4

$$i(z_i + z_1) + (i - i_1)z_2 = E \qquad ....(7.3.1)$$

$$i(z_i + z_1) + i_1 z_2 = E \qquad ....(7.3.2)$$

Eliminating i from above equations we have

$$i_L = \frac{Ez_2}{z_L(z_i + z_1 + z_2) + z_2(z_i + z_1)} = \frac{Ez_2 \big/ (z_1 + z_1 + z_2)}{z_2(z_i + z_1) \big/ (z_i + z_1 + z_2)} = \frac{V_{Th}}{z_L + z_{Th}} \qquad .....(7./3.3)$$

Which is same as the yield of the Thevenin's theorem thus proving the theorem.

Procedure of Theveniirs Theorm:

Find the open circuit voltage at the terminals, Voc.

Find the Thevenin's equivalent resistance, R_{TH} at the terminals when all independent sources are zero.

❖ Replacing independent voltage sources by short circuit

❖ Replacing independent current sources by open circuit

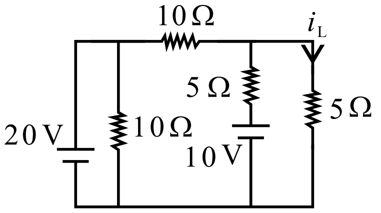


Fig 7.5

Reconnect the load to the Thevenin equivalent circuit

Example-1 : Find the current i_L through 5Ω resistor in the adjoining circuit.\

Solution :

$$20 = 10i + 5i + 10, \quad \text{i.e. } i = \frac{10}{15} = \frac{2}{3}, \text{ So } V_{Th} = 5.\frac{2}{3} + 10 = \frac{40}{3}\text{V}$$

From Fig (7.6)

$$\text{and } R_{Th5} = \frac{5 \times 10}{15} = \frac{10}{3}\Omega \quad \text{So } i_L = \frac{\frac{40}{3}}{\frac{10}{3} + 5} = \frac{40}{25} = \frac{8}{5} = 1.6\text{A}$$

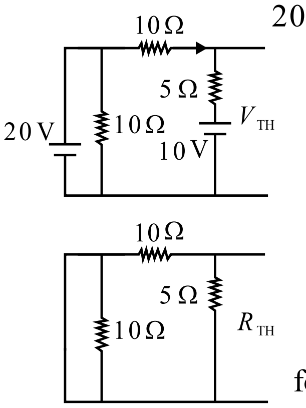


Fig 7.6

Example-2 : The four arms of a Wheatstone bridge have the following resistances: AB=100Ω, BC=10Ω, CD=4Ω, DA=50Ω. A galvanometer of 20Ω resistance is connected across BD. Use thevenin's theorem to compute the current through the galvanometer when a p.d of 10V is maintained across AC.

Solution :

We proceed to apply Thevenin's theorem in the Circuit.

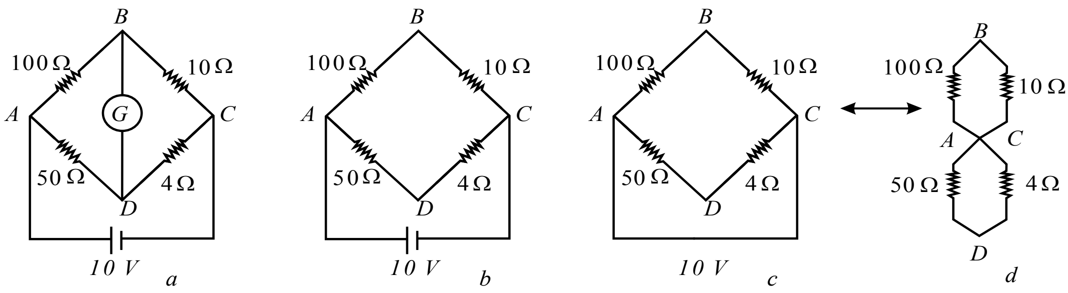


Fig 7.7

b) Galvanometer is removed and the $V_{BD}=V_{Th}$ is calculated

$$V_{Th}=\left(\frac{10}{110}.10-\frac{10}{54}4\right)=10\left(\frac{1}{11}-\frac{2}{27}\right)=0.168\text{V}$$

d) The voltage source is short-circuited and R_{Th} is calculated across BD.

$$R_{Th}=\frac{100\times 10}{100+10}+\frac{50\times 4}{50+4}=12.79\Omega$$

So the current through the galvanometer $i_G=\dfrac{V_{Th}}{R_{Th}+R_L}=\dfrac{0.168}{12.79+20}=5\text{mA}$

7.4 Norton's Theorem

The theorem states that any two terminal linear bilateral network with energy sources and resistances can be replaced by current source with current I_N (Norton's current), obtained by short circuiting the chosen terminals and an resistance R_N (Norton's resistance), in parallel to it obtained across the terminal by replacing the energy sources by their respective internal resistances.

Fig (7.8) gives an illustrative presentation of Norton's theorem.

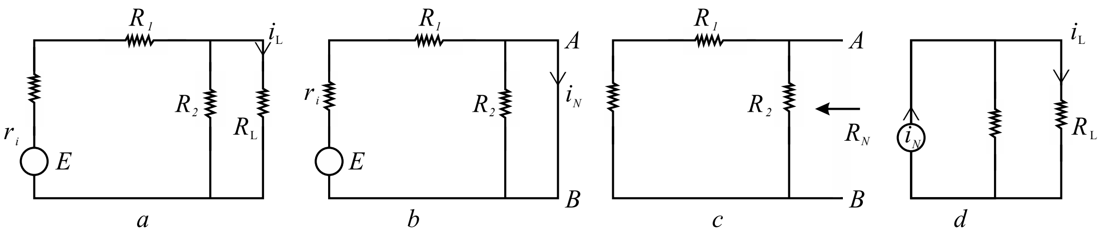


Fig. 7.8

a) The circuit for which i_L is to be determined; r_i is the internal impedance of the voltage generator.

b) Load R_L is short-circuited and short circuit current called Norton's current is calculated through the terminal AB $i_N = \dfrac{E}{r_i + R_i}$

c) E is short-circuited and Norton's impedance $R_N = \dfrac{R_2(r_i + R_1)}{r_i + R_1 + R_2)}$ is calculated across AB.

d) Gives the Norton's equivalent circuit with current $i_i = \dfrac{R_N i_N}{R_L + R_N}$

Proof of Norton's theorem (using Kirchhoff's laws)

The proof is similar to the proof of Thevenin's theorem. Here in Norton's theorem we have

$$i_L = \frac{R_N i_N}{R_L + R_N} = \frac{\dfrac{R_2(r_i + R_1)}{(r_i + R_1 + R_2)} \dfrac{E}{(r_i + R_1)}}{R_L + \dfrac{r_2(r_i + R_1)}{(r_i + R_1 + R_2)}} = \frac{ER_2}{R_1(r_i + R_1 + R_2)_L + R_2(r_i + R_1)},$$

which is same as the result from Kirchhoff's laws, thus proves the Norton's Theorem.

Procedure of Norton's Theorm :

1. Find the short circuit current at the terminals, I_{SC}.
2. Find Thevenin's equivalent resistance, R_{TH} (as before).
3. Reconnect the load to Norton's equivalent circuit.

Example

Let us solve Example-1 using Norton's theorem.

From Kirchhoff's laws,

$20V - 10i_i = 0$, *or* $i_1 = 2$ A *and* $10V - 5i_2 = 0 \Rightarrow i_2 = 2A$. So $i_N = i_i + i_2 = 4A$

From the adjoining figure (7.9)

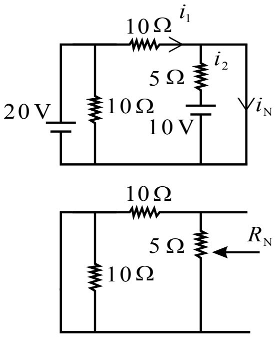


Fig 7.9

$$R_N = 5 \| 10 = \frac{50}{15} = \frac{10}{3}\Omega$$

$$i_i = \frac{4 \times 10/3}{5 + 10/3} = 1.6A$$

7.5 Superposition Theorem

The theorem states that any linear bilateral network with several energy sources, the current/voltage in any element will be the algebraic sum of contribution from each source

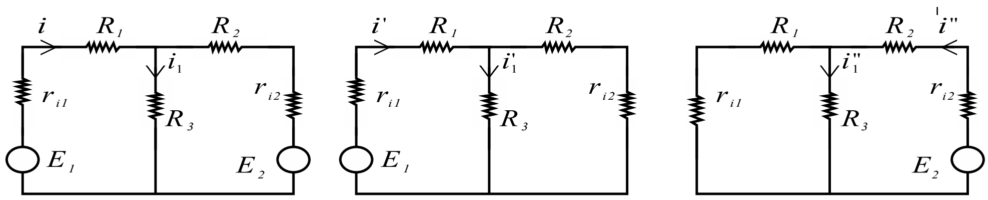


Fig.10(a) *Fig. 10(b)* *Fig. 10(c)*

replacing the other sources by their respective internal impedances.

Fig. 7.10

Proof :

The Fig. 7.10(a) hows a circuit to analyze, Fig. 7.10(b) and Fig. 7.10(c) are the circuits for analyzing the problem using Superposition theorem.

$$E_i = (r_{i1} + R_1)i' + R_3 i_1' \qquad ......(7.5.1)$$

Then from Fig. 7.5(b)

$$0=(R_3+r_{i2})(i-i_1')-i_2'R_2=-(R_3+r_{i2})i'+(R_3+r_{i2}+R_2)i_1' \qquad ......(7.5.2)$$

Solving the above two equations by Cramer's rule, we have

$$i_1'=\frac{1}{\Delta}\begin{vmatrix} r_n \to r_{i1}+R_1 & E_1 \\ -(r_{i2}+R_3 & 0 \end{vmatrix}$$

Where $\Delta=\begin{vmatrix} r_{i1}+R_1 & R_2 \\ -(r_{i2}+R_3 & r_{i2}+R_2+R_3 \end{vmatrix}$

Similarly from Fig. 7.10(c) $i_1''=\frac{1}{\Delta}\begin{vmatrix} r_{i1}+R_1) & 0 \\ -(r_{i2}+R_3) & E_2 \end{vmatrix}$

So $i_1'+i_2''=\frac{1}{\Delta}\begin{vmatrix} r_{i1}+R_1) & E_1 \\ -(r_{i2}+R_3) & E_2 \end{vmatrix}$ is the total current through the Resistance R_2, according to superposition theorem.

Now we find out the same solution using Kirchhoff's laws. Please refer Fig. 7.10(a) From Kirchhoff's laws we have,

$$E_1 = (R_1 + r_{i1})i + i_1R_2$$

$$E_2 = (r_{i2} + R_3)(i_1 - i) + i_1R_2 = -(r_{i2} + R_3)i + (r_{i1} + R_3 + R_2)i_1$$

From above two equations eleminating i we have,

$$i_1=\frac{1}{\Delta}\begin{vmatrix} (r_{i1}+R_1) & E_1 \\ -(r_{i2}+R_3) & E_2 \end{vmatrix}$$

which shows $i_1 = i_1'+i_1''$ proving superposition theorem from Kirchhoff's laws.

Procedure :

1. Dependent source are Never deactivated (always active)
2. When an independent voltage source is deactivated, it is set to zero, replaced by short circuit
3. When an independent current source is deactivated, it is set to zero, replaced by open circuit

Example

Please refer Example-1 : The efm source 10V is short circuited. Then equivalent resistance about 20V emf generator is

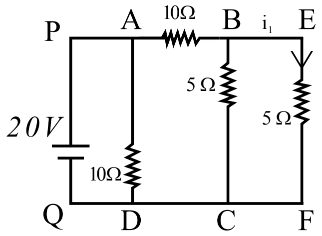
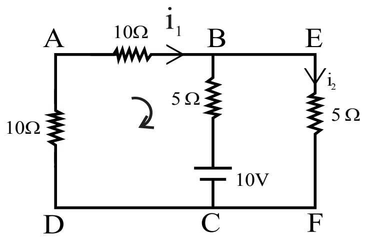


Fig. 7.11(a)

$$R_1 = 10 \,||\, (10 + 5 \,||\, 5) = 10 \,||\, 12.5 = \frac{125}{22.5}\Omega$$

$$\text{So, } i_1 = \frac{10}{125} \times \frac{10}{12.5} \times \frac{1}{2} = .72\text{A}$$

Fig. 7.11(b)



From the principle of superposition theorem we now, activate the source 10 volt and deactivate 20 volt source, and find out the current in the are EF having 5Ω.

Applying Kirchoff's law in ABCD network,

$$-10i_1 - 5(i_1 - i_2) - 10 - 10i_1 = 0 \qquad \text{...... (a)}$$

And, applying Kirchoff's law in the BEFC network,

$$-5i_2 + 10 - 5(i_2 - i_1) = 0 \qquad \text{.......(b)}$$

Solving equations (a) and (b) we find

$$i_2 = \cdot 88\text{A}$$

Total current in the arm EF having 5Ω resistance

$$i = i_1 + i_2$$

$$i = \cdot 72 + \cdot 88 = 1 \cdot 6\text{A}$$

7.6 Maximum Power Transfer Theorem

The theorem states that in a linear bilateral with resistances the energy supplied to the load resistance reaches maximum when the load resistance equals the energy source resistance.

Proof : Consider the circuit, Fig. (7.12) with a generator of emf E having internal resistance r_i connected to external load R_L. Current through the load resistance

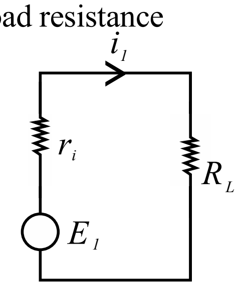


Fig. 7.12

$$R_L, i_L = E/(r_i + R_L)$$

So the power delivered to the load resistance,

$$P = i_L^2 R_L = \frac{E^2 R_L}{(r_i + R_L)^2}$$

For maximum power $\frac{dP}{dR_L} = 0 \Rightarrow E^2 \left[\frac{1}{(r_i + R_L)^2} - \frac{2R_L}{(r_i + R_L)^3}\right]$

So, $r_i = R_L$, which proves the theorem.

The percentage of power efficiency of a circuit is defined as

$$\eta\% = \frac{\text{power delevered to the external load}}{\text{total power consumed in the circuit}} \times 100 = \frac{R_L}{r_i + R_L} \times 100$$

Thus efficiency of a circuit becomes 50% when the circuit deliver maximum power to the external load.

The variation of $\eta\%$ with R_L is shown in Fig (7.6.3)

The variation of power dissipated across the load with P across the load resistance R_L is shown in Fig. (7.6.4)

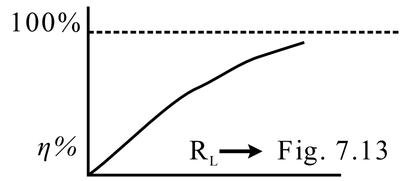


Fig. 7.13

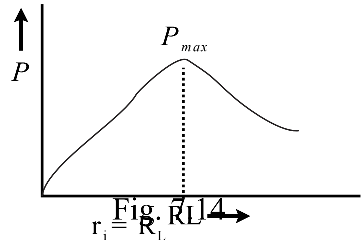


Fig. 7.14

Example :

1) In the Fig. 7.1.5 Find the value of R for maximum power transfer in 3Ω resistance.

Solution :

Using maximum power theorem we have for this condition

$$\frac{R \times 6}{R+6} = 3, \qquad R = 6\Omega$$

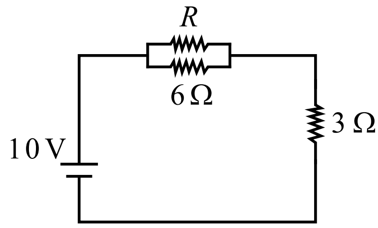


Fig 7.15

7.7 Reciprocity theorem

The theorem states that in a linear bilateral network with sources and resistances, if the source of emf in one mesh produce a current in another mesh then the transfer of the emf source to the second mesh keeping its internal resistance in the previous position will produce same current in the first mesh.

Proof:

The Fig. 7.16(a) shows the given network and the Fig. 7.16(b) shows the changed

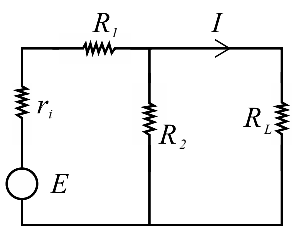


Fig.7.16(a)

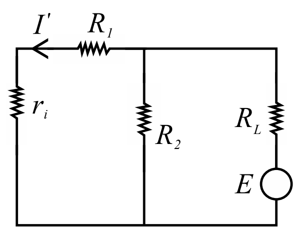


Fig.7.16(b)

network following Reciprocity theorem.

Consider Fig. 7.16(a), here

$$I = \frac{E}{(r_i + R_1) + R_2 \,||\, R_L} \times \frac{R_2}{R_2 + R_L} = \frac{ER_2}{(r_i + R_1)(R_2 + R_L) + R_2 R_L}$$

Now from Fiq. 7.16(b) we see that

$$I' = \frac{E}{R_L + R_2 \,||\, (r_i + R_1)} \times \frac{R_2}{R_2 + (R_1 + r_i)} = \frac{ER_2}{(r_i + R_1)(R_2 + R_L) + R_2 R_L}$$

So, $I = I'$, that proves Reciprocity theorem.

Proceedure :

Conditions to be met for the application of reciprocity theorem :

(i) The circuit must have a single source.

(ii) Initial conditions are assumed to he absent in the circuit

(iii) Dependent sources are excluded even if they are linear

(iv) When the positions of source and response are interchanged, their directions should he marked same as in the original circuit.

Example :

1. Two cells of emf e_1 and $e_2 (e_1 > e_2)$ and internal resistances r_1 and r_2 are connected in parallel to the ends of a wire of resistance R. Find the current through R using i) Thevenin's theorem ii) Norton's theorem iii) Superposition theorem.

Solution :

The circuit diagram is shown in the Fig. 7.17.

i) i_R using Thevenin's theorem.

Step-1 To find V_{Th}

Remove R and find the open circuit p.d V_{Th}

Current through the loop

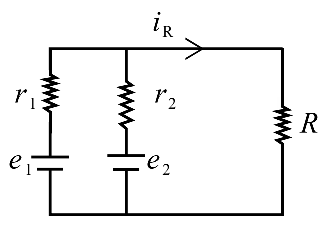


Fig 7.17

$i = \dfrac{e_1 - e_2}{r_1 + r_2}$, So the potential across the ab $V_{Th} = e_1 - ir_1 = \dfrac{e_1 r_2 + e_2 r_1}{r_1 + r_2}$

Step-2 To find R_{Th}

Short circuit the emf source and find the resistance across ab.

$$R_{Th} = r_1 || r_2 = \frac{r_1 r_2}{r_1 + r_2}$$

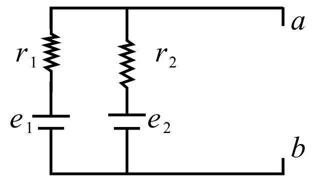


Fig 7.18

So the current through R

$$i_R = \frac{V_{Th}}{R_{Th} + R} = \frac{\dfrac{e_1 r_2 + e_2 r_1}{r_1 + r_2}}{\dfrac{r_1 r_2}{r_1 + r_2} + R} = \frac{e_1 r_2 + e_2 r_1}{r_1 r_2 + r_1 R + r_2 R}$$

ii) i_R using Norton's theorem.

Step-1 To find i_N. KT

Short circuit the resistance R and find the current through ab.

Norton's current $i_N = \dfrac{e_1}{r_1} + \dfrac{e_2}{r_2} = \dfrac{e_1 r_2 + e_2 r_1}{r_1 r_2}$

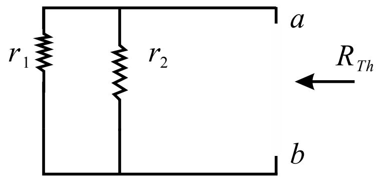
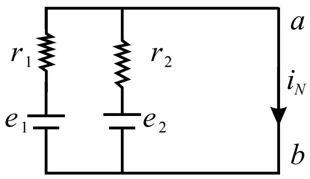


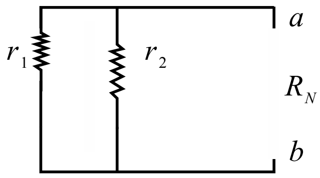
Fig 7.19

Step-2 To find R_N

Step 1



Step 2



Here the Norton's resistance $R_N = \dfrac{r_1 r_2}{r_1 + r_2}$

Fig. 7.20

So current through the resistance R

$$i_R = \frac{R_N i_N}{R+R_N} = \frac{\frac{r_1 r_2}{r_1+r_2}\,\frac{e_1 r_2 + e_2 r_1}{r_1 r_2}}{R+\frac{r_1 r_2}{r_1+r_2}} = \frac{e_1 r_2 + e_2 r_1}{r_1 r_2 + r_1 R + r_2 R}$$

iii) i_R using Super Position theorem.

Step-1 Short circuit e_1 and evaluate i_1.

$$i_1 = \frac{e_2}{r_2+\frac{r_1 R}{r_1+R}}\,\frac{r_1}{r_1+R} = \frac{e_2 r_1}{r_1 r_2 + r_1 R + r_2 R}$$

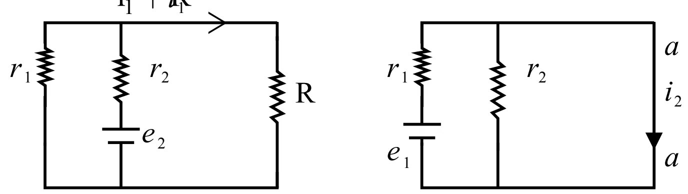


Fig 7.21 (a) Fig 7.21 (b)

Fig. 7.21

Step-2 Short circuit e_2 and evaluate i_2

$$i_2 = \frac{e_1}{r_1+\frac{r_2 R}{r_2+R}}\,\frac{r_2}{r_2+R} = \frac{e_1 r_2}{r_1 r_2 + r_1 R + r_2 R}$$

The total current through R, $i_R = i_1 + i_2 = \dfrac{e_1 r_2 + e_2 r_1}{r_1 r_2 + r_1 R + r_2 R}$

7.8 Summary

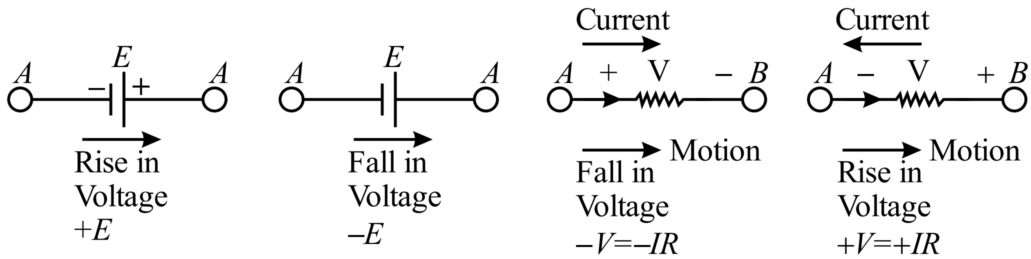
(1) Network circuit Theory is a useful procedure to analyze and simply the complex circuit in different configuration. Equivalent circuits are drawn by applying different theorems like Thevenin, Norton superposition and Reciprocity theorems. Also we have shown that

Thevenins and Nortons circuit theory are all equivalent in analyzing circuit having bilinear port. Different types of problems have been discussed using Kirchoff's laws, nodal theory and mesh network to simply Thevenin and Nortons circuit. We have seen the condition for maximum power transfer from source to the load when source impedence is equal to the load impedence.

(2) Determination of Voltage Sign : In applying Kirchhoff's laws to specific problems, particular attention should be paid to the algebraic signs of voltage drops and e.m.fs., otherwise results will come out to be wrong. Following sign conventions is suggested :

(a) Sign of Battery E.M.F.

A rise in voltage should be given a + ve sign and a fall in voltage a –ve sign. Keeping this in mind, it is clear that as we go from the –ve terminal of a battery to its +ve terminal



(Fig. 7.2.3), there is a rise in potential, hence this voltage should be given a +ve sign. If, on the other hand, we go from +ve terminal to –ve terminal, then there is a fall in potential, hence this voltage should be preceded.

Limitation of Thevenin's and Norton's Theorem

(1) These theorems used only in the analysis of linear circuits

(2) The power dissipation of the Thevenin's equivalent is not identical to the power dissipation of the real system. Super position theorem limitation—the requisite of linearity indicates that this theorem is only applicable to determine voltage and current but not power.

7.9 Review Questions and Answers

1. What are the steps to follow Thevenin's Theorem?

Ans. See section (7.3) for answer.

2. What are the steps to follow Norton's Theorem?

Ans. See section (7.4) for answer.

3. Convert the voltage source of figure to a current source

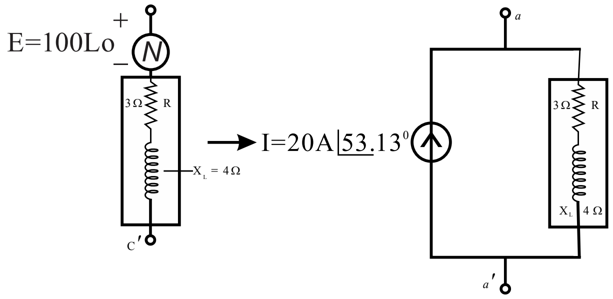


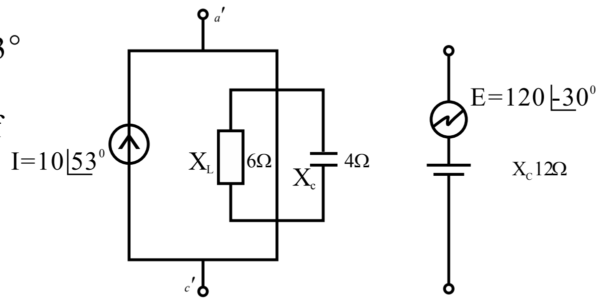
Fig. 7.24

Solution :

$$I=\frac{E}{Z}=\frac{100\angle 0}{5\angle 53.13°}=20\angle -53.13°$$

4. Convert the current source of figure to a voltage source.

Solution : **Fig. 7.25**



$$z=\frac{z_c \times z_L}{z_c+z_L}=\frac{(4\angle -90°)(6\angle 90°)}{-j4+j6}=-j12=12\Omega\angle -90°$$

$E = IZ = (10\angle 60°)(12\angle -90°) = 120\angle -30°$

7.10 Problems and Solution

1. In the diagram given in Fig 7.2.6 determine the Norton's equivalent source current and resistance with respect to the terminals a,b.

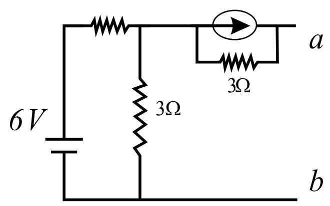
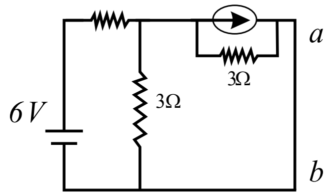


Fig. 7.26(a)

Solution :

Step-1 : Short circuit ab, then the short circuit current



$$i_N = \frac{6}{3+3||3} \times \frac{3}{3+3} + 2 = \frac{3}{4.5} + 2 = 0.67 + 2 = 2.67\text{A}$$

Step-2 : Short circuit 6V and measure the resistance across ab.

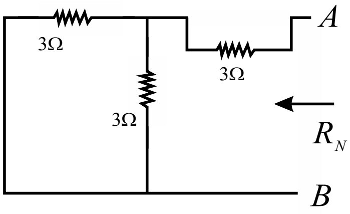


Fig. 7.26(b)

R_N = 3+3||3=4.5Ω

Fig. 7.26(c)

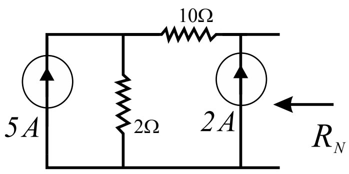
Problem-2 :

Determine the current through 5W r esistor in Fig. 7.27(a)

Solution :

Step-1 : Short circuit 5Ω resistance and find i_N.

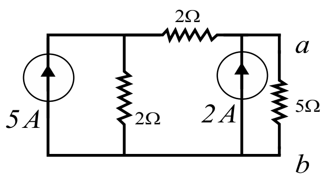
Fig. 7.27(b)



$$i_N = 2 + 5 \times \frac{2}{12} = \frac{17}{6}$$

Step-2 : Open the 5Ω resistance and find R_N.

Fig. 7.27C



Equivalent circuit of Fig. 7.24(d) is

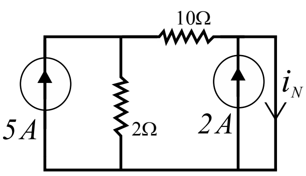
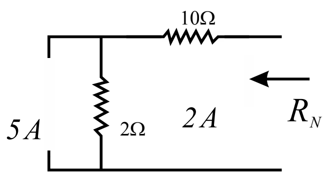
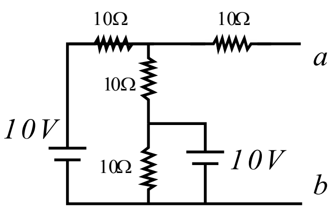


Fig. 7.24(d)

So R_N = 12Ω



So current through 5Ω resistance $i_5 \dfrac{\frac{17}{6} \times 12}{12+5} = 2\text{A}$

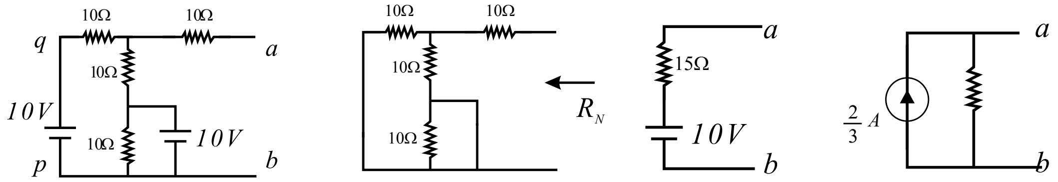


Problem-3 :

Find the i) Thevenin's and ii) Norton's equivalent circuit of the adjoining Fig. 7.25 between a and b.

Fig. 7.28

Solution :



Since no current flows through pq, so V_{ab} = 10V

R_{Th} = 10 + 10||10 =10+5=15Ω

Thevenin's equivalent circuit

Norton's equivalent circuit

Fig. 7.29

Problem-4

Find the current through 5kΩ resistance in the circuit in Fig. 7.30(a) using Thevenin's theorem.

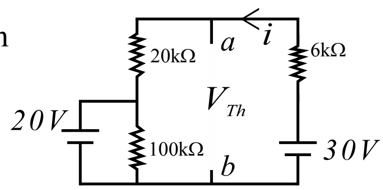


Fig. 7.30(a)

Solution :

Step-1 : Open circuit 5kΩ and find V_{Th}.

From the adjoining circuit

$$30-20=6i+20i \text{ or } i=\frac{10}{26}\text{mA}$$

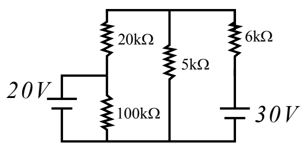


Fig.7.30(b)

$$\text{So, } V_{Th}=30-6\times\frac{10}{26}=30-\frac{30}{13}=\frac{30\times 12}{13}\text{V}$$

Step-2 : Replace the emf generator by their respective internal resistances and calculate R_{Th}.

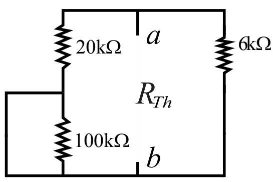


Fig. 7.30(c)

From adjoining circuit

$$R_{Th}=6||20=\frac{6\times 20}{26}=\frac{60}{13}\text{k}\Omega$$

$$\text{So, } i_5 = \frac{V_{Th}}{R_{Th}+5} = \frac{\frac{30\times12}{13}}{\frac{60}{13}+5} = \frac{30\times12}{125} = 2.88\,\text{mA}$$

Problem 5 : Solve for the power delivered to 20Ω resistor in the circuit diagram shown in Fig. 7.31(a). All the resistances are in ohms.

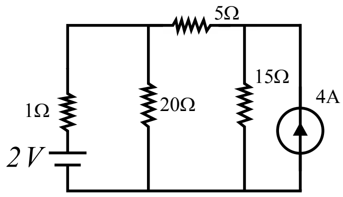


Fig. 7.31(a)

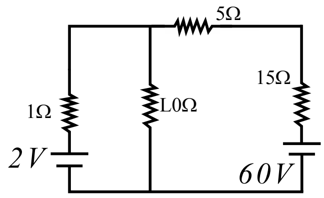


Fig. 7.31(b)

Solution : 4A source and its parallel resistance can be converted into a voltage source (15×4)=60 volt in series with a resistance as shown in Fig. 7.31(a).

Now using superposition theorem to find the current through the 20Ω resistor, when 60V source is removed, the total resistance as seen by 2V baliery is 1 + 2011 (15+5)11Ω.

The battery current is $= \frac{2}{11}$A. At point P, the current is divided into two parls. The current passing throng 20Ω is $I_1 = \frac{2}{11}\Big/(20+20) \times 20 = \frac{1}{11}\text{A} = .09\text{A}$

When 2 Volt battery removed, the resistance as seen by the battery 60 Volt is 20.95. The current from the battery is 28693A $\approx$ 2.87A

This current divides at point A, the current through the 10Ω is $I_2 = \frac{2.87}{20+1} \times 1 = 0.14\text{A}$

Total current flows through 20Ω is = $I_1 + I_2$

= 0.09 + 0.14 = .23A

Unit 8 ❑ Electrical Circuits

Structure

8.1 Objectives

You will know from this unit—

- All the parameters of AC—voltage and current, its average value, root mean square value (RMS)
- Application of complex number
- Kirchoff's Laws

- Behaviour of Resistance, inductance and Capacitor
- Series LCR Circuit, its unique resonant properties
- Parallel LCR Circuit its unique resonant properties and uses.

8.2 Introduction

Alternating current (AC) is an electric current which periodically reverses direction and changes magnitude continuously with time in contrast to direct current (DC) which is unidirectronal. The most common form of wareform of alternating current in most electric power circuits is a sinewave, whose positive half period corresponds with positive direction of the current and vice versa.

We will study the physical properties of resistor R, inductor L, and Capacitor C under the impact of AC, and the nature of current flow and wattless current. Use of complex number is essential as it is convenient to represent and calculate both AC, signals and impedance. Two dimension length and angle allows as to calculate amplitude and phase together, and keep them consistent. Unique properties of combinational circuit like series LCR and parallel LCR, culminating to the concept of 'Band Width', Quality Factor, which is of practical importance in physics and engineering will also be studied in detail.

8.3 Alternatring Current and Its Characteristics

Alternating Current (AC) or voltages is current or voltage which periodically reverses direction and changes its magnitude continuously with time. The most common form of AC is sinusoid. Even, if it is nonsinusoid it can be resolved into many sinusoid by Fourier Transform. For symmetric AC its average over a complete cycle is zero.

The most common form of generation of AC works on the principle of Faraday's law of electromagnetic induction. Whenever a coil is rotated in a uniform magnetic field about an axis perpendicular to the field, the magnetic flux linked with the coil changes and an induced emf is set up across its ends. A pure sinusoidal voltage is represented in Fig. 8.1 can be written as

$$v(t) = V_0 \sin wt \qquad ......(8.3.1)$$

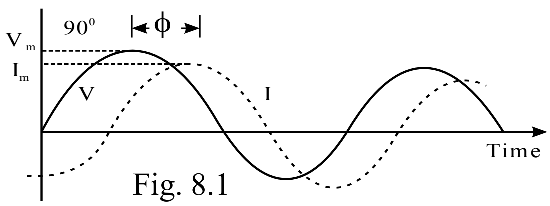


Fig. 8.1

Here v(t) is the instanceous value of the voltage and V_0 its amplitude, w its angular frequency time period T is relatred to frequency $f = \frac{w}{2\pi} = \frac{1}{T}$. When this alternating voltage applied in a circuit, current flows through it is given by

$$i(t) = I_0 \sin(wt + \phi) \qquad ....(8.3.2)$$

2. RMS value of AC Waveforms : The rms value of an alternating current is given by that steady (dc) current which when flowing through a given circuit for a given time produces the same joule heat as produced by the alternating current when flowing through the same circuit for the same time.

The root mean square value of an alternating current with period T is given by

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)} \ \ dt \qquad .... (8.3.5)$$

It can be related to the heating effect of ac. Total Joule heat produced by ac in a resistance R overf a time peiod T is

$$\int_0^T i^2(t).R.dt = I_{rms}^2 RT \qquad ....(8.3.6)$$

For a simple sinusoidal current $i = I_0$ sinwt the rms value is

$$I_{rms}^2 = \frac{1}{T}\int_0^T I_0^2 \sin^2 wtdt = \frac{I_0^2}{2}$$

$$\text{So, } I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{\text{Peak value of current}}{\sqrt{2}} \qquad ....(8.3.7)$$

Similarly for a sinusoidal voltage $V_{rms} = \frac{V_0}{\sqrt{2}}$. I_{rms} or V_{rms} is a measurable value as all measuring instruments based on the heating effect of current and calibrated accordingly. The peak value of the domestic AC, mains supply of rms voltage 220V is 220V2 = 311V. Because of this high peak value of 311V from AC mains is more shocking than same value of DC supply of 220V.

3. Form Factor : The ratio of the rms value to the average value over a half cycle of a periodic function is defined to be the form factor of the periodic waveform. For a sinusoidal ac, the form factor is given by.

Form factor $K_f = \dfrac{I_0/\sqrt{2}}{2I_0/\pi} = 1.11$ ......(8.3.8)

From factor gives an idea about the wave shape. Any deviation in the value of K_f from 1.11 indicates deviation from sinusoidal nature.

4. Power in AC circuits : According to Joule heat energy generated is proportional to the square of current flow. In alternating current rate of electrical energy P spent in a circuit varies with time and though, at any instant rate energy spent being the product of voltage and current, but in reality average P is effective parameter. for electrical energy spent. So the average value of P is

$$\overline{P} = \frac{1}{T}\int_0^T v(t)i(t)dt$$

where v(t) = V_0 sinwt and i(t) I_0 sin(wt+ϕ), here ϕ is the phase angle.

Now, $$\overline{P} = \frac{V_0 I_0}{T}\int_0^T \sin wt.\sin(wt+\phi)dt$$

or, $$\overline{P} = \frac{V_0}{\sqrt{2}}.\frac{I_0}{\sqrt{2}} \cos\phi$$

So, $\overline{P} = V_{rms} \times I_{rms} \cos\phi$ ......(8.3.9)

The term cosϕ is known as power factor. The product $V_{rms}I_{rms}$ does not, except for the case cosϕ = 0, gives the true power dissipated in the circuit as does the product in d.c. circuit. Here V_{rms} I_{rms} is called apparent power. While $\overline{P}$ gives the real power in the circuit. So we have,

Real power = Apparent power × Power factor ......(8.3.10)

In A.C. circuit instantaneous power is given by p(t) = i(t).v(t) and i(t) have same sign. The positive value of p(t) imdicates that the source of A.C. supply is delivering energy to the circuit. Again when v(t) and i(t) have opposite sign, implying that the source is receiving energy from the circuit when phase angle $\phi=\pi/_2$ or 90°, so no power is dissipated in the circuit. The current flowing in such curcuit is called wattless current. It will be seen latter that when current flowing through pure inductor or pure capacitor is a wattless current.

5. Peak factor : The ratio of the peak value to the rms value of any ac waveform is called its peak factor. For a pure sinusoidal ac,

$$\text{peak factor } (k_p) = \frac{I_0}{I_0/\sqrt{2}} = \sqrt{2} \qquad \text{........(8.3.11)}$$

Since waves of same rms value may have different peak values a knowledge of peak factor is necessary to get an idea if peak value. Again a knowledge of peak value is important while testing dielectric insulation or hysteresis loss.

8.4 Representation of sinusoidal ac by complex numbers

Two main reasons that make the use of complex numbers suitable to model AC, circuits and many other sinewave phenomena in several branches of science and technology are described below.

1. The AC signals are characterised by a magnitude and phase that are, respectively very similar to the modulus and argument of complex numbers.

2. The basic operations such as addition subtraction multiplication and division of complex number are easier to carryout.

We know that when a vector is multiplied by –1, though its magnitude remains unaltered, but the direction changes in opposite direction i.e. 180°. As $-1 = \sqrt{-1} \times \sqrt{-1}$, so we multiply or operate $\sqrt{-1}$ two times in 180° polar angle. Hence operating $\sqrt{-1}$ one time on a vector it will rotate in 90° in anticlocwise direction.

In a two dimensional co-ordinate system (XY), let X-axis represents real number and imaginary number along Y-axis (Fig. 8.2). It a vector $\vec{A}$ is along the positive X-direction then $\bar{\bar{A}}$ vector quantity will be perpendicular to $\vec{A}$. Denoting $\vec{A}$ as $j\vec{B}$, then the resultant of $\vec{A}$ and $j\vec{B}$ will be another vector $\vec{P}$ as shown in Fig. 8.3

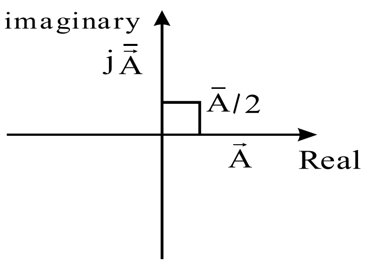


Fig. 8.2

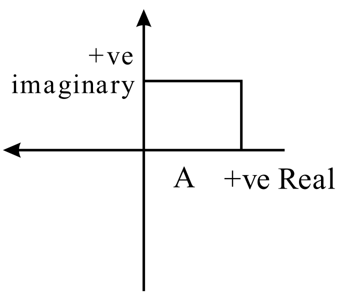


Fig. 8.3

Clearly for A + jB, its magnitude XO = $\sqrt{A^2 + B^2}$ which is real quantity and the phase angle with respect to real axis $\phi = \tan^{-1B}/_A$ is inclined.

So if $A = X_0\cos\phi$ and $B = X_0\sin\phi$ are taken, then A and B are replaced by X and ϕ real numbers and consequently A + jB is expressed as complex number

$$A + jB = X_0\cos\phi + jX_0\sin\phi = X_0e^{j\phi}.$$

Here $X_0 = \sqrt{A^2 + B^2}$ is magnitude of A + jB and $e^{j\phi}$ is its phase term.

In case of alternating current, if $\upsilon = V_0\sin wt$ or $V_0\cos wt$ is taken, then they are real and imaginary part of $v = V_0e^{jwt}$. So v or i can be explained in complex plane whoe vector disposition or phasor $X = A + jB = X_0e^{jwt}$, the phasor X will be rotating with time at the angular speed w and its value X_0 remains unchanged.

8.4.1 Impedence and Reactance

In AC, circuit, current flow, which faces resistance is called impedence or reactance when voltage v and current flow i are expressed in complex number, the impedance can be expressed by the ratio of v and i let $v = Ve^{jwt}$ and $i = I_0e^{v(wt+\phi)}$, then by the analogy of ohms law in DC circuit, we have v = zi, here z is called the imp-edance or reactance, so, $z = \frac{v}{i} = \frac{V_0}{I_0}e^{-j\phi t}$, the real part of z is called resistance R and imaginary part is called reactance X.

So, $\qquad z = R + jX$

From the above equation, we get

$$R = \frac{V_0}{I_0}\cos\phi$$

Earlier in this unit, we have shown that $P = V_{rms}I_{rms}\cos\phi$

$$= I_{rms}^{\ 2}|z|\cos\phi$$

$$= I_{rms}^{\ 2}R$$

Comparing this, above average power dissipated with the Joule's law of heat in steady flow of current, we can say that real part of z is to denoted as resistance of the circuit.

8.5 Kirchoff's Laws

Kirchoff's laws of AC circuits are as follows :

1. The phasor sum of the currents at any point in the circuit is zero.
2. The phasor sum of the voltages around any closed loop is zero.

The current and voltage equations are derived in the same way as in DC circuits. The algebric simplifications of phasor quantities is no different from that of DC quantities until numerical quantities are introduced.

For KVL, let v_1, v_2........v_n be the voltages around a closed loop. Then

$$v_1 + v_2 + \text{..........} + v_n = 0 \qquad \text{......(8.5.1)}$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that, Equation (8.5.1) becomes

$$V_{01} \cos (wt + \theta_1) + V_{02}\cos(wt + \theta_2) + \text{..........} +$$
$$\text{............} + V_{0n} \cos(wt+\theta n) = 0 \qquad \text{............(8.5.2)}$$

This can be written as,

$$\mathrm{Re}(V_{01}e^{j\theta i}e^{jwt}) + \mathrm{Re}(V_{02}e^{j\theta 2}e^{jwt})$$
$$\text{.....}+\mathrm{Re}(V_{0n}e^{j\theta n}e^{jwt}) = 0 \qquad \text{......(8.5.3)}$$

$$\text{or,}\quad \mathrm{Re}[V_{01}e^{j\theta 1}+V_{02}e^{j\theta 2} + \text{........}+ V_{on}e^{j\theta n}]e^{jwt} = 0$$

If we let $V_s = V_{0s}e^{j\theta_s}$, then

$$\mathrm{Re}[(V_1 + V_2 + V_3 +\text{......}+V_n e^{jwt}] = 0 \qquad \text{......(8.5.4)}$$

Since $e^{jwt} \neq 0$

$$V_1 + V_2 + \text{......} + V_n = 0 \qquad \text{.......(8.5.6)}$$

indicating that kirchoff's voltage law holds for phasors.

In the same way, we can show that Kirchoff's current law holds for phasors. If we let i_1, i_2........i_n be the current entering or leaving closed circuit in a network at time t, then

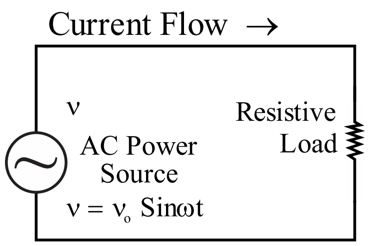
$$i_1 + i_2 + \text{..........} + i_n = 0 \qquad \text{......(8.6.7)}$$

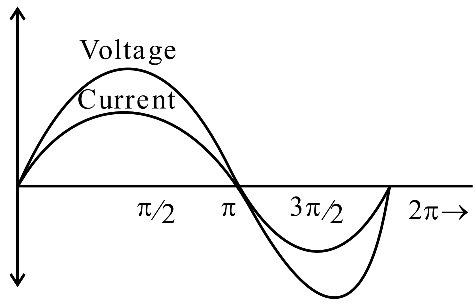
If I_1, I_2.........I_n are the phasor forms of the sinusoids i_1, i_2.........i_n then

$$I_1 + I_2 + ... + I_n = 0 \qquad ....(8.5.8)$$

which is Kirchoff's current law in the AC domain.

8.6 AC Responses of a Resistance, An Inductance and a Capacitance





1. Resistance R

Fig. 8.4 **Fig. 8.5**

In the above AC circuit potential difference across the resistor / conductor is v(t) = i(t)R, here R is the resistance of the conductor, which follows from ohms law. As R is a real quantity, so, v(t) and i(t) will be in the same phase, and they are both indicated in the phasor diagram along the positive real number axis by two straight line (Fig. 8.5).

2. Pure inductance 'L'

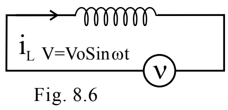


Fig. 8.6

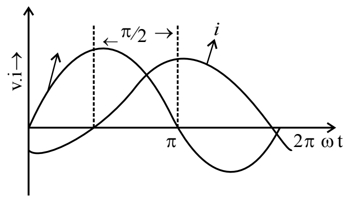


Fig. 8.6 Pure inductive circuit

Alternating voltage v(t) is applied across inductor L. We know that when a time varying current flows through an inductor, a back emf is produced if the coil is having inductance L, the back induced emf is $-\frac{Ldi}{dt}$, which mitigates the supplied AC voltage source v(t) so the voltage equation of the circuit $v - \frac{Ldi}{dt} = 0$, To find solution, we apply complex analysis as discussed earlier in this unit. Let $i(t) = I_0 e^{j\omega t}$ is the instantaneous current in the circuit, then

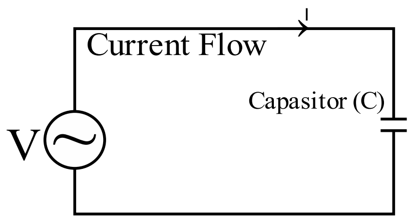
$$v(t) = \frac{Ldi}{dt} = j\omega L I_0 e^{j\omega t} = j\omega L I_0 \qquad \text{........(8.6.1)}$$

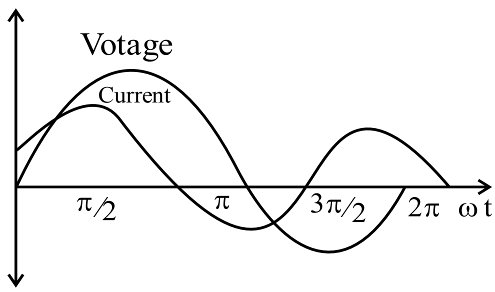
so the impedence of the circuit $z = j\omega L$, which is an imaginary number, so $z = \omega L e^{j\pi/2}$

Now we get,

$$v(t) = \omega L I_0 e^{j(\omega t+\pi/2} \qquad \text{.......(8.6.2)}$$

Taking V_0 as the maximum potential difference, then $V_0 = \omega L I_0$ and alternating voltage source is ahead of the current flow in phase by $\pi/2$. In Fig 8.7 phasor representation of v and i is shown.





(3) Pure Capacitance

Fig. 8.8 **Fig. 8.9**

When an alternating voltage is supplied across capacitor charged first in one direction and then in the opposite direction. Let $v(t) = V_0 \sin\omega t$

Instantaneous charge at the capacitor plate

$$Q_t = C\, V_0 \sin\omega t \qquad \text{.......(8.6.3)}$$

So, the charging current at any instant

$$i(t) = \frac{dQ_t}{dt} = \omega\varepsilon V_0 \cos\omega t$$

$$= \frac{V_0}{y_{\omega c}} \sin(\omega t + \pi/2)$$

or, $i(t) = I_0 \sin(\omega t + \pi/2)$ .....(8.6.4)

Here I_0 is the peak current given hy = $\frac{I_o}{\frac{1}{\omega c}}$ obviously, current i(t) through a pure capacitor leads over the applied voltage by 90°. Hence power dissipated by the circuit containing pure capacitar is zero as power factor $\cos\phi = 0$.

Rate of heat dissipation in the circuit at any instant

$$P(t) = v(t)\, i(t)$$

$$= \frac{v_0^2}{\omega L} \cos\omega t \; \cos(\omega t - \pi/2)$$

$$= \frac{V_0^2}{2\omega L} \sin 2\omega t$$

Average rate of dissipation of heat energy is

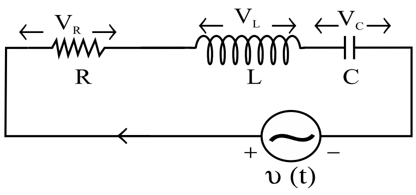
$$\overline{P} = \frac{V_0^2}{2\omega L} \frac{1}{T} \int_0^T \sin^2 \omega t dt = 0$$

So, there is no dissipation of energy in a circuit containing capacitance. Continually at an interval of $T/4$ time source current electrical energy is transformed to magnetic energy returns it to the source as electrical energy. Such mutual exchange of energy is non-dissipated.

8.7 Series LCR Circuit

Resistance R, inductance L and a capacitance C are connected in series with an AC source. So source emf. $V = V_R + V_L + V_C$ and separately resistance are as follows R the resistance, impendance of L is $Z_L = j\omega L$ and impedence of C, $z_c = -j/\omega c$

Fig. 8.10



So, at any instant relation between the voltage and current is given by

$$v = (R + j\omega L - {}^{j}/_{\omega c})\ i = [R + J(\omega L - {}^{1}/_{\omega c})]i \qquad ......(8.7.1)$$

Here circuit resistance is R and reactance is $X = (\omega L - {}^{1}/_{\omega c})$

$$\text{Hence } i = \frac{v}{|z_0|}e^{-j\phi} \qquad ....(8.7.2)$$

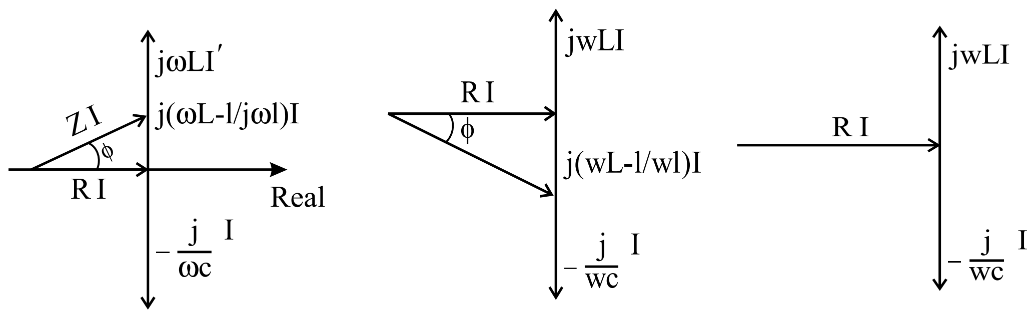
Now the total impedence of the circuit is

$$|z_0| = \frac{1}{\sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega c}\right)^2\right]}} \qquad ......(8.7.3)$$

Phase relation between v and i, $\phi = \tan^{-1}\frac{X}{R}$ as shown in Fig. 8.11(a).

Let source emf at any instant is $v(t) = V_0\cos\omega t$ is real part of $V_0e^{j\omega t}$. So the current flow at any instant is the real part of $i(t) = \frac{V_0}{|z_0|}e^{j(\omega t-\phi)}$ where

$$\phi = \tan^{-1}\frac{\omega L - \frac{1}{\omega c}}{R} \qquad ......(8.7.4)$$



Phasor diagram (a) $\omega L > {}^{1}/_{\omega c}$, (b) $\omega L < ({}^{1}/_{\omega c})$ and (c) $\omega L = {}^{1}/_{\omega c}$

Fig. 8.11

From eq^n (8.7.4) it appears that phase angle is not alway positive, so the phase relationship of v(t) and i(t) varies with the circuit parameter.

i) $[\omega L - {}^{1}/_{\omega c}] > 0$ is positive then current flow legs behind source emf v(t).

ii) $[\omega L - {}^{1}/_{\omega c}] < 0$, ϕ is negative, v(t) lags behind i(t).

iii) $\omega L = 1/_{\omega c}$ then, $\phi = 0$, v(t) and i(t) are in the same phase.

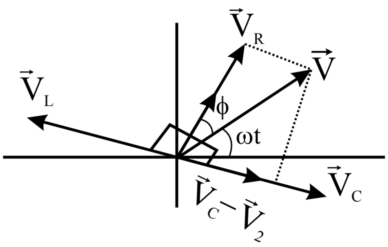
Under third condition, mentioned above, impendence of the circuit becomes $z_0 = R$, that is it is purely resistive and lowest value maximum current flows though the circuit (with maximum value) under this condition, frequency of AC source becomes

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}} \qquad ......(8.7.5)$$

This state of AC circuit is called resonant and ω_0 is resonant frequency. In reality resonant circuit behaves as purely resistive even in the presence of L and C.

8.7.1 Phasor relation graph

Here, phasor such as v(t), Ri, $j\omega Li$ and ${}^{-ji}/_{\omega c}$ are shown in the graph with direction,



in Fig.12

Fig. 8.12

8.7.2 Different types of Resonance in Series LCR

We have shown that under resonant condition, maximum current flows through the circuit, when maximum root mean square current us given by

$$I_{rms} = \frac{V_{rms}}{R} \qquad ......(8.7.6)$$

Root mean square current, other than resonant condition is given by

$$I_{rms} = \frac{V_{rms}}{\sqrt{\left[R^2 + \left(\omega L - {}^{1}\!/_{\omega c}\right)^2\right]}} \qquad .....(8.7.7)$$

and phase angle $\phi = \tan^{-1}\dfrac{\omega L - \frac{1}{\omega c}}{R}$.

In reality current resonant condition can achieved either by changing the frequency of the source or changing value of capacitance may bring different values of I_{rms}. Also it can brought a voltage resonant condition by other than series resonant state in the circuit.

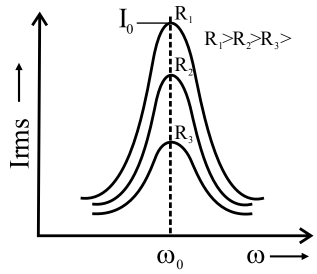


Fig. 8.13

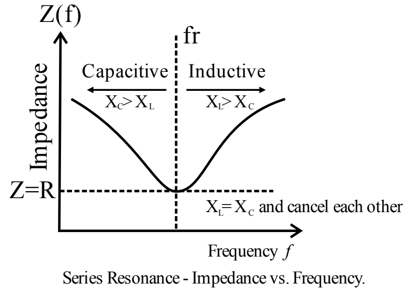


Fig. 8.14

8.7.3 Quality Factor and Shapness of Resonance in Series LCR Circuit :

The sharpness of resonance relates to the reapdity of the fall in current on eigherside of the resonance frequency. The current falls to a very low values depending on the sharpness of the resonant. The smaller the value of the resistance the greater the current at resonance and the shasper the resonance. Behaviour of R in circuit in forming sharper resonance is shown in Fig. 8.13.

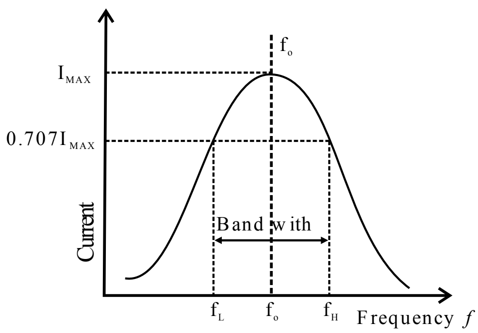


Fig. 8.15

If we now reduce or increase the frequency until the average power absorbed by the resistor R in series with resonance circuit is half that of its maximum value at resonance, we produce two frequency points called the half power points which are –3d B down from maximum taking OdB as the maximum current reference the point corresponding to the lower frequency at the half power point is called the lower cutoff frequency, denoted as f_L the point corresponding to the upper frequency for half power point called higher frequency cult of f_H. The difference $f_H - f_L$ is called the bandwith (Fig. 8.15).

At the half power point, impedence $z_0 = \sqrt{2}R$.

$$\text{or,} \quad \sqrt{R^2 + \left(\omega L - \frac{1}{\omega c}\right)^2} = \sqrt{2}R$$

$$\text{So,} \quad \left(\omega L - \frac{1}{\omega c}\right) = \pm R$$

$$\omega^2 LC \pm \omega RC - 1 = 0$$

Only the positive roots are acceptable, Rools are

$$\omega_1 = \frac{\sqrt{(R^2C^2 + 4LC)} - RC}{2LC}$$

$$\text{and} \quad \omega_2 = \frac{\sqrt{(R^2C^2 + 4LC)} + RC}{2LC} \qquad \text{.......(8.7.8)}$$

The bandwidth is

$$\omega_2 - \omega_1 = \frac{R}{L} = \frac{\omega_0 R}{\omega_0 L} = \frac{\omega_0}{Q} = \frac{1}{\omega_0 CR}$$

$$\text{or,} \quad Q = \frac{\omega_0}{\omega_2 - \omega_1} \qquad \text{.......(8.7.9)}$$

This equation relates the Q to the bandwidth. Sharpness of resonance increases with the increase in Q. Quality factor increases with the decrease in R, as there is no change in resonant frequency. Graph (Fig. 8.13) shows the dependency of sharpness with variation in R. The circuit can store energy in the form of magneticfield or electrical energy across the condenser. The performance efficiency is also given by the Q-factor

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}}$$

$$= \frac{2\pi \times \frac{1}{2} L I_0^2}{I_r^2 RT}$$

where T is the time period, and $I_r = \frac{I_0}{\sqrt{2}}$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \qquad \text{......(8.7.10)}$$

Another term, used in resonant circuit is selectivity. It is defined is its ability to respond more readily to signals than to signals of other frequencies. This response becomes progressively weaker as the frequency departs from the resonant frequency it is mathematically defined as

$$\text{Selectivity} = \frac{\text{Bandwidth}}{\text{Resonant frequency}} = \frac{f_2 - f_1}{f_0} \qquad \text{......(8.7.11)}$$

8.7.4 Voltage Resonance in Series LCR Circuit :

Here we will study the behaviour of changing C and L in a series LCR circuit and show other types resonance. Let us take potential diffference across the capacitor.

$$V_c = \frac{I_{rms}}{\omega C} \qquad \text{.........(8.7.12)}$$

But, $$I_{rms} = \frac{V_{rms}}{\sqrt{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}}$$

Therefore, $$V_c = \frac{V_{rms}}{\sqrt{\left[\omega^2 C^2 R^2 + \left(\omega^2 LC - 1\right)^2\right]}} \qquad \text{.......(8.7.13)}$$

The variation of |Vc| with ω is shown in the figure

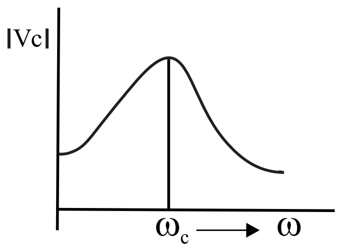


Fig. 8.16

Therefore, for the maximum $|V_c|$ we must show

$$\frac{d}{d\omega}\left[\omega^2R^2C^2+(\omega^2LC-1)^2\right]=0$$

After differentiation

$$\omega=\omega_C=\left[\frac{1}{LC}\left(1-\frac{R^2C}{2L}\right)\right]^{1/2}$$

$$\text{or,}\quad \omega_C=\omega_0\left[1-\frac{1}{2Q^2}\right]^{1/2} \qquad \text{.......(8.7.14)}$$

Equation (8.14) shows that $\omega_c < \omega_0$

The reason is that current peaks at resonance and capacitor voltage is the product of current and $(^1/_{\omega c})$ which decreases with increasing frequency. $\omega_c \simeq \omega_0$ when Q is very large.

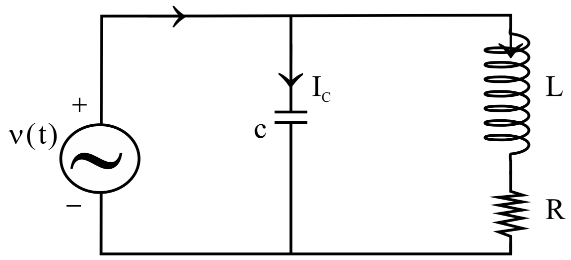
8.8 Parallel Resonant Circuit

When an alternating voltage source is connected to an inducetor having small resistance combinedly in parallel with a capacitor, which is shown in Fig. 8.17. Two current component from the source is divided into two branches of parallel resonant circuit. These are phasor current I_{RL} and I_C so total phasor current $I = I_{RL} + I_C$ and impedences $Z_c = {}^{-j}/\omega_C$ and $z_{RL} = R + j\omega L$ source phasor voltage $V = V_{RL} + V_C$. When R is small, I_L and I_C will be almost 180° out of phase, since the capacitor current I_C leads the source voltage V by 90° and while the inductor current I_L lags the source voltage V by nearly 90°. At resonance V and I are in phase.

$$I=\frac{V}{R+J\omega L}+j\omega CV=\frac{V}{z} \qquad \text{......(8.8.1)}$$

where z is the complex impedence of the circuit.

Fig. 8.17



$$\frac{1}{z} = \frac{1}{r + j\omega L} + j\omega C = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$= \frac{R}{R^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$

The source voltage V and the source current I will be in phase when the imaginary part of $^1/_z$ varishes. Then

$$\omega C = \frac{\omega L}{R^2 + \omega^2 L^2} \quad \text{or,} \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \omega_0 \qquad .....(8.8.3)$$

the source voltage and current are in phase, power factor is unity, this condition of the circuit is said as parallel resonant frequency. For ω_0 to be real, the condition must be

$$\frac{R^2}{L^2} < {}^1\!/_{LC} \quad \text{or,} \quad R < \sqrt{{}^L\!/_C} \qquad ....(8.8.4)$$

The impedence of the circuit is

$$R_d = \frac{R^2 + \omega_0^2 L^2}{R} = R + \frac{L^2}{R}\left(\frac{1}{LC} - {}^{R^2}\!/_{L^2}\right)$$

$$R_d = \frac{L}{CR} \qquad ......(8.8.5)$$

R_d is called dynamic resistance.

At the resonance peak current from the supply is called make up current and is given by

$$I_{op} = \frac{V_0}{R_d} \qquad ......(8.8.6)$$

The cupacitor current at parallel resonance is

$$I_{cp} = \frac{V_0}{{}^1\!/_{w_0 c}} \qquad ....(8.8.7)$$

$$\text{So,} \ \frac{I_{cp}}{I_{op}} = \omega_0 C R d = \frac{\omega_0 L}{R} \qquad ....(8.8.8)$$

Q of the circuit is defined as the ratio of the capacitor current to the line current at parallel resonance. Hence,

$$Q = \frac{I_{cp}}{I_{op}} = \frac{\omega_0 L}{R} \qquad ......(8.8.9)$$

the capacitor current I_{cp} is much large than the supply current due to high value of Q. At resorce, capacitor or inductor current is Q times the supply current. Such highly magnified current of parallel resonance can be compared to magnified voltage of series resonance circuit.

From equation (8.8.2) the impedence z or admittance Y can be written as

$$\frac{1}{z} = \frac{R}{R^2 + \omega^2 L^2} + j\left[\omega C - \frac{\omega L}{(R^2 + \omega^2 L^2}\right]$$

or, $$\left|\frac{1}{z}\right| = |Y|^2 = \frac{R^2}{(R^2 + \omega^2 L^2)^2} + \left[\omega C - \frac{\omega L}{(R^2 + \omega^2 L^2}\right]^2 \qquad .....(8.8.10)$$

Maximum impedence or minimum admittance at a frequency Q_m is given by

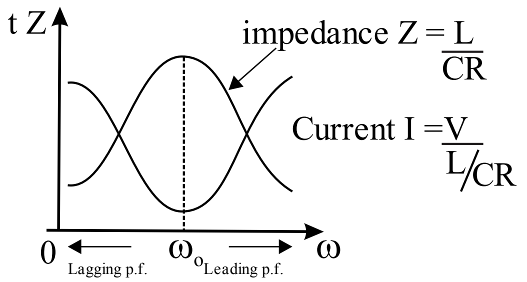
$$\frac{d}{d\omega}|Y|^2 = 0$$

which gives $$\omega_m = \left[\frac{1}{LC}\left(\frac{2CR}{L} + 1\right)^{1/2} - \frac{R^2}{L^2}\right]^{1/2} \qquad ......(8.8.11)$$

comparing eq^h (8.8.3) and (8.8.11) we find that $\omega_m > \omega_0$, i.e. maximum impedence is achieved at higher frequency than the resonant frequency ω_0. If Q is large as R is low, then we can assume that $\omega_n \approx \omega_0$. This circuit is called anti resonant because current is minimum at resonance which is in contrast to the series resonant current. Sometimes it is called rejector circuit.

8.8.1 Selectivity of Parallel Resonant Circuit :

(Fig. 8.18) shows the variation of z with frequency. For $R \to 0$, then $\omega_m \approx \omega_0$, which is the condition for high Q circuit. Let $\omega_0 \pm \Delta\omega_0$ then if the impedence of the circuit z becomes $\frac{z_0}{\sqrt{2}}$, we say that resonance has become very sharp. Then select band of frequency which is allowed blocked or filtered, which is the condition for high selectivity of parallel resonant circuit.



ure: Resonance Curve for Parallel Resonance

Fig. 8.18

From equition (8.8.2)

$$\frac{1}{z} = \frac{1}{R + j\omega L} + j\omega c. \qquad .....(8.8.11)$$

when R → 0, The above equation can be written as

$$z = \frac{j\omega L}{(1-\omega^2 LC) + J\omega CR} \qquad ......(8.8.12)$$

At resonance $\omega = \omega_0$ and $z = z_0 = \frac{L}{CR}$

From equation (8.8.12) $z = \dfrac{z_0}{1 - \frac{j}{R}\left(\omega L - \frac{1}{\omega c}\right)}$

$$\text{or,} \qquad |Z| = \frac{z_0}{\sqrt{1 + \frac{1}{R^2}\left(\omega L - \frac{1}{\omega c}\right)^2}} \qquad .....(8.8.13)$$

For high selectivity we have chosen $z = \frac{z_0}{\sqrt{2}}$ (8.8.13) becomes after simplification.

$$\omega L - \frac{1}{\omega c} = \pm R$$

$$\text{or,} \qquad \omega^2 \mp \frac{R}{L}\omega - \frac{1}{LC} = 0 \qquad ....(8.8.14)$$

Writing $\omega_0^2 \approx \frac{1}{LC}$ for assuming low value of R

$$\omega_1 \mp \frac{\omega_0}{Q}\omega - \omega_0^2 = 0 \qquad ......(8.8.15)$$

Real roots are,

$$\omega_1 = \omega_0\left[-\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}\right]$$

and $$\omega_2 = \omega_0\left[\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}\right] \qquad ......(8.8.16)$$

So the band width is

$$\Delta\omega = \omega_2 - \omega_1 = \frac{\omega_0}{Q} \qquad ......(8.8.17)$$

From equation (8.8.17) we conclude that for high selectivity, preferred choice remains for high value of Q.

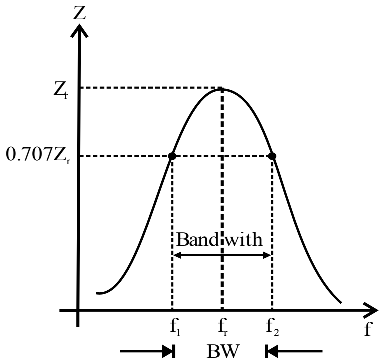


Fig. 8.8.4

Parallel Resonance In LCR Circuit.

Fig. Resonance Curve for Parallel Resonance

Fig.

From the eq, we have

$$V_C = \frac{V_{rms}}{\sqrt{\left[\omega^2C^2R^2 + (\omega^2Lc - 1)^2\right]}}$$

The value of C for the maximum value of V_c can be found in the following way

Let $x = V[\omega^2C^2R^2 + (\omega^2Lc-1)^2]$

$$\frac{dx}{dc} = 2\omega^2R^2C + 2\omega^2L(\omega^2Lc - 1) = 0$$

$$C = C_0 = \frac{L}{R^2 + \omega^2 L^2}$$

So, this resonance can be spoiled with the variation of R. When C = C_0, maximum of value of Vc is

$$V_{c(max)} = \frac{V_{rms}\sqrt{(R^2 + \omega^2 L^2)}}{R}$$

8.9 Summary

1. In AC circuits, it is seen that impedance of reactive components, like inductor or capacitor is expressed in terms of imaginary number, or, phasor qualysis of impedence of inductance by $jL\omega$ or for capacitor by $\frac{1}{j\omega C}$. Total electrical energy dissipated in this component is zero.

2. The frequency at which impedence becomes minimum in series LCR circuit is called resonant frequency. At resonant, angular frequency $\omega_0 = \frac{1}{\sqrt{(LC)}}$ and quality factor is $Q = \frac{\omega_0 L}{R}$.

8.10 Review Questions and Answers

1. Define bandwidth.

Ans. It is defined as the breadth of the resonant curve upto frequency at which the power in the circuit is half if its maximum value. The difference between two half power frequencies is called the bandwidth.

2. Define selectivity.

Ans. The selectivity of a RLC circuit is the ability of the circuit to respond to a certain frequency and discriminate against all other frequencies. If the band of frequencies to be selected or rejected is narrow, the quality factor of the resonant circuit must be high

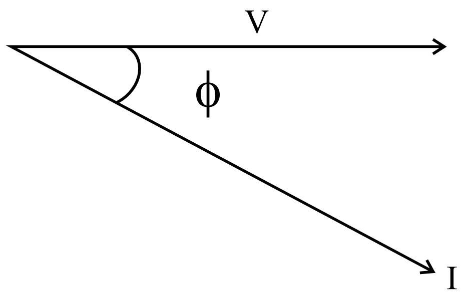
$$\text{Selectivity} = \frac{\text{Bandwidth}}{\text{Resonant frequency}} = \frac{f_H - f_L}{f_0}$$

3. Define phasor and phase angle.

Ans. A sinusoidal wareform can be represented in terms of phasor. A phasor is a vector with definite magnitude and direction. From the phasor, the sinusoidal waveform can be constructed. Phase angle is the angular measurement that specifies the position of the alternating quantity relative to a reference.

4. Define power factor.

Ans. Power factor is defined as the cosine of the angle between voltage and current. If ϕ is the angle between voltage and current that $\cos\phi$ is called as the power factor. Other definition is the ratio between real power and apparent power.



5. Define Apparent power and power factor.

Ans. The apparent power (inVA) is the product of the rms values of voltage and current, $S = I_{rms} V_{rms}$. The power factor is the cosine of the phase difference voltage and current. It is also the cosine of the load impedence. Power factor = cosf. The power factor is leading when current loads voltage (capacitative) and lagging when current lags voltage (inductive load)

8.11 Problems and Solutions

1. A coil takes a current of 1A from 6v dc supply, when connectexd to a 120v 5 OHZ supply the current is 10A. Calculate the resistance, impedence, inductive reactance and inductance of the coil.

Solution : Resistance $R = \dfrac{\text{dc voltage}}{\text{dc current}} = \dfrac{1}{6} = 6\Omega$

$$\text{impedence} = \frac{\text{ac voltage}}{\text{ac current}} = \frac{120}{10} = 12\Omega$$

Since $z = \sqrt{R^2 + X_L^2}$

So, inductive reactance $X_L = \sqrt{(z^2 - R^2} = 10.39\Omega$

Again, $X_L = 2\pi fL$

$$\text{inductance } L = \frac{X_L}{2\pi f} = \frac{10.39}{21 \times 50} = 33.1\text{mH}$$

2. A coil of resistance 5Ω and inductance 120mH in series with 100 mF capacitor, is connected to a 220V, 50 Hz supply. Calculate (a) the current flowing, (b) the phase difference between the current and supply voltage (c) the voltage across the coil and (d) the voltage across the capacitor.

Solution : Circuit diagram is shown in the fig

$$X_L = 2\pi fL = 2\pi(50)(120 \times 10^{-3}) = 37.7\Omega$$

$$X_C = \frac{1}{2\pi fc} = \frac{1}{2\pi(50)(100 \times 10^{-6})} = 31.83\Omega$$

Since X_L is greater than X_C the circuit is inductive

$$X_L - X_C = 5.87\Omega$$

Inpedence $z = \sqrt{\left[R^2 + (X_L - X_C)^2\right]} = 7.71\Omega$

(a) Current $I = \frac{v}{z} = \frac{220}{7.71} = 28.53\text{A}$

(b) Phase angle $\phi = \arctan\left(\frac{X_L - X_C}{R}\right) = \arctan\frac{5.87}{5} = 49°35'$

(c) Impedence of the coil = $\sqrt{(R^2 + X_L^2)}$ =38.03Ω

(d) Voltage across the coil.

$= IZ_{coil} = 38.03 \times 28.53 = 1085\text{V}$

Phase angle of the coil

$$= \text{arctan } \frac{X_L}{R} = \text{arctan}^{-1}\frac{37.7}{5} = 82°45'$$

Voltage across the capacitor V_C

$$= IX_C = 28.53 \times 31.83 = 90\delta.10V$$

3. A coil of inductance 0.1 H and resistance 30W is connectred in parallel with a 10 μF capacitor across a 50V, variable frequency AC supply calculate (a) the resonant frequency, (b) the dynamic resistance, (c) the current at resonance and (d) the circuit Q-factor at resonance.

Solution :

(a) Parallel resonant frequency,

$$f_r = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - {R^2}/{L^2}\right)} = \frac{1}{2\pi}\sqrt{\left[\frac{1}{0.1\times(10\times10^{-6}} - \frac{30^2}{(0.1)^2}\right]} = 152 \text{ Hz}$$

(b) Dynamic resistance Rd $= \dfrac{L}{RC} = \dfrac{0.1}{30\times(10\times10^{-6})} = 333.33\Omega$

(c) Current at resonance $I_r = \dfrac{50}{333.33} = 0.15A$

(d) Circuit Q-factor at resonance $= \dfrac{2\pi frL}{R} = 3.183$

Capacitor current at resonance $I_C = \dfrac{V}{X_C}$

$$= \frac{V}{\left(\frac{1}{2\pi f_r c}\right)} = 0.477\text{Amp}$$

Hence Q factor $= \dfrac{I_C}{I_r} = \dfrac{0.477}{0.15} = 3.183$

4. A complex voltage (20 + j.O)V is applied to a series LR Circuit of complex impedence (1+√3j)Ω. Calculate the power factor and the power consumed by the circuit.

Solution :

Complex current

$$I = \frac{20 + j10}{1 + 3j} = \frac{20}{\sqrt{(3)^2 + 1^2}} = 10e^{-j\phi}$$

where $\tan\phi = \sqrt{3}$

So the current lags behind the emf by an angle $\phi = \tan^{-1}\sqrt{3} = 60°$

$\therefore$ Power factor $\cos\phi = \cos 60° = 0.5$

Power consumed $P = VI\cos\phi$

$= 20 \times 10 \times 5$

$= 100$ watt.

5. An electric lam p which runs at 100V and 10A current is connected across 220 Volt 50 cycle AC main. Calculate the value of the choke to be connected in series for the lamps safety.

Solution :

Here the resistance of the circuit or lamp $R = \frac{V}{I} = \frac{100}{10} = 10\Omega$

Impedance $z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{10^2 + (2\pi \times 50)^2 L^2}$

We have $z = \frac{220}{10} = 22$

So, $L = \frac{\sqrt{22^2 - 10^2}}{(2\pi \times 50)^2} = 0.0623\text{H}.$

Unit 9 □ Ballistic Galvanometer

Structure

9.1 Objectives

After the completing this chapter the learner will understand -

- The construction and operation of electrical measuring instrument- the galvanometer.
- How a ballastic Galvanometer is used to mesure charge
- How a ballastic Galvanometer can be converte it into dead-beat galvanometer to mesure current and voltage.
- What is CDR and its rate to current a galvanometer from ballastic to dead-beat and V.C.V.S
- The current sensitivity, voltage sensitivity and charge sensitivity of galvanometer and their radiation.

9.2 Introduction

A ballastic galvanometer is a type of instrument, commonly a miror galvanometer, unlike a current- measureing glavanometer. The moving part has a large moment of inertia and hence giving it a long period of ocillation period. The glavanometer works on the principle of permanent magnet moving coil. The force is generated on the coil, due to Lorentz Force law. "Due to interaction of fluxes, the pointer in the meter or miror is deflected. As it is the deffected different torqes to make the pointer stop at its steady state motion. The different torques are deffeching control torques and damping torquesis almost zero. For that reason.it is called a ballastic galvanometer. It is really an integrator measuring the quantity of charge or discharge throught it. It can be used ad voltmeter and ammeter.

9.3 Introduction

The fig. 9.1 shows a line diagram of a moving coil galvanometer. It consists of

a) A rectangular frame with insulated copper wire round on it.

b) The frame is suspended symametrically with a thin phosphor bronze fibre with its end connected to the one end of copper wire.

c) C is a nonconducting or conducting core palced symametrically inside the frame.

d) N-S represent concave magnetic poles placed co-contri at the mid point of the gap.

e) The bottom end of the copper wire connected to a phosphor-bronze thread to measure the deflection of coil using lamp and, scale arrangement.

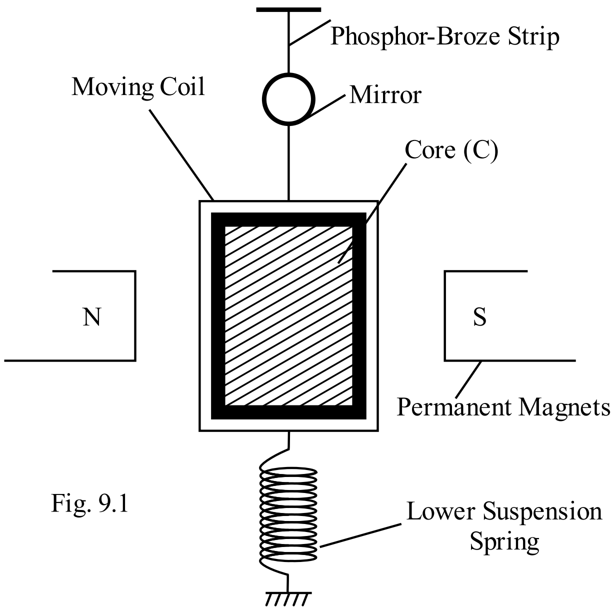


Fig. 9.1

f) The small mirror M, attached to due suspended phosphor-bronze thread to measure the deflection of coil using lamp and scale arrangement.

Theory-

We consider the coil c to have n number of turns with area A (Vertical lenght l and horizental width b, A=lb) suspended in an uniform magnetic field B dq be the amount of charge flow through the coil in time dt at an instant t, so i = dq/dt at instant t.

The torque on the coil O→$\tau = \vec{M} \times \vec{B}$ (where M= niA).

Taking the magnitude of torque we have

$$\tau = M\,B = niAB = nAB\frac{dq}{dt} \text{...(9.3.1)}$$

If I is the moment of inertia of the coil with core along the axis of suspention, then $I\frac{d\omega}{dt} = nAB\frac{dq}{dt}$ [w being the angular speed at instant t]

$$\text{or } I\,d\omega = nABdq$$

If we consider the moment of inertia of the coil-core system is sufficiently high such that the coil-core system does not move during the passege of charge q_0, Then intergrating the eqn. (9.2) we have

$$I\omega_0 = n\,A\,B\,q_0 \text{...(9.3.2)}$$

If there is no disipitive force, then from conservation of energy we can write,

$$\tfrac{1}{2}\,I\,\omega_0^2 = \tfrac{1}{2}\,c\theta_0^2 \text{...(9.3.3)}$$

Where θ_0 is the angular amplitude of first throw.

After migration of charge the equation of motion of the coil at an instant t will be

$$I\,\frac{d^2\theta}{dt^2} = -C\theta, \text{ where C represents the torque}$$

per unit turist (or torsional rigidity) of the suspension fiber. So the time period of osciallation of the coil

$$T = 2\pi\sqrt{I/c} \quad \text{or } I = \frac{CT^2}{4\pi} \text{..............(9.5)}$$

Eliminiting ω_0 from eqns. 9.3 & 9.4 and putting the value of I from eqn. 9.5, we have

$$q_0 = \frac{T}{2\pi}\;\frac{C}{nAB}\;\theta_0 \text{ ....(9.3.4)}$$

Theory of moving Coil galvanometer with damping forces.

When a coil moves in a galvanometer two damping forces play important role to oppose the motion.

1. Mechanical damping force, mostly due to the air friction which at low angular velocity of coil can be taken to be proportinal to the angular speed.

$\omega \left(= \frac{d\theta}{dt}\right)$, say $F_a = \frac{d\theta}{dt}$

2. Electromagnetic damping force, due to rotation of coil in magnetic field. To proceed with the calculation of e.m damping, please refer Fig. 9.2. Due to rotation of coil through dq in a inform magnetic field B, that lies always in the plane of the coil, the vertical section passes through the area d.s = $2\left(\frac{b}{2} d\theta \text{ x } l\right)$ = bldθ.

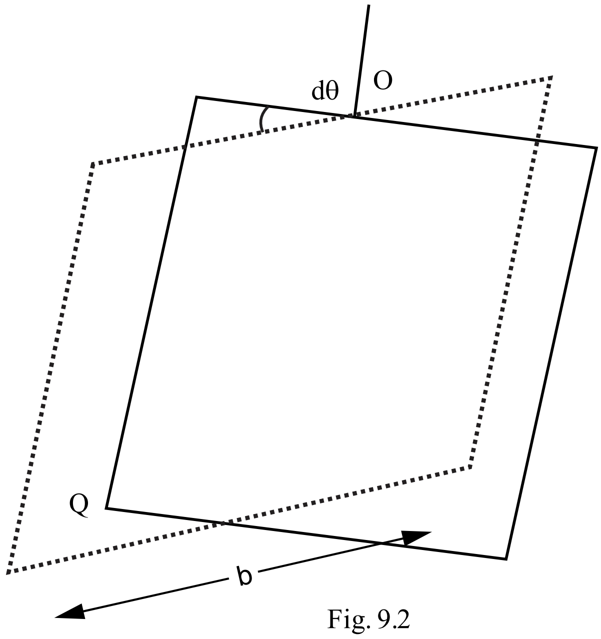


Fig. 9.2

so the number of time of forces intercepted by a single turn of coil

= bldθ B.

so the rate of change of magnetic lines through the coil of n turns

$$= nblB \frac{d\theta}{dt} = nAB \frac{d\theta}{dt}$$

$$= G \frac{d\theta}{dt}$$

Where G = nAB is called golavanometer constant. So the induced end generated in the coil to oppose the motion

$$e = G \frac{d\theta}{dt} \quad \text{......(9.3.5)}$$

so the induced current

$$i_d = \frac{G}{R} \frac{d\theta}{dt} \quad \text{......(9.3.6)}$$

Where R = R_G (Galvanometer resistance) + Re (External resistance in the circuit)

The opposing torque developed due to this induced current

$$= ni_d AB = \frac{G^2}{R} \frac{d\theta}{dt} \quad \text{......(9.3.7)}$$

so the equation of motion of coil considering the damping force takes the form,

$$I\frac{d^2\theta}{dt^2} = C\theta - a\frac{d\theta}{dt} - \frac{G^2}{R}\frac{d\theta}{dt} = C\theta - (a + \frac{G^2}{R})\frac{d\theta}{dt}$$

or $$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + \omega_o^2\theta = 0 \quad \text{...(9.3.8)}$$

wherw $2b = (a + G^2/R)/I$ and $\omega_o^2 = C/I$

To seek for a solution we put $q = Ae^{\alpha t}$, then we have from eqn.(9.3.8)

$$\alpha 2 + 2b\alpha + \omega_o^2 = O \text{ or } \alpha = b \pm \sqrt{b^2 - \omega_o^2}$$

Thus the solution of eqn. (9.3.8) coils down to

$$\theta = A\,e^{(-b+\sqrt{b^2-\omega_o^2})\,t} + B\,e^{(-b-\sqrt{b^2-\omega_o^2})\,t}$$

$$= e^{-bt}\left[A\,e^{\sqrt{b^2-\omega_o^2}\,t} + B\,\bar{e}^{\sqrt{b^2-\omega_o^2}\,t}\right] \text{..................................(9.3.9)}$$

Now for b>ωo, The first term exponentially decreases with time, the first term within braces increase exponentilly with time but less effectively then the outside term; the second term descreses exponentially with time. Hence the motion is a damped non-oscillatory motion. Similar logic leads that when $b = \omega_o$ motion is non-oscillatory and is known critically damped motion.

When $b<\omega_o$ the eqn. (9.3.9) can be written as

$$\theta = e^{-bt}\left(A_1 e^{+i\sqrt{\omega_o^2 - b^2}\,t} + A_2 e^{-i\sqrt{\omega_o^2 - b^2}\,t}\right)$$

or $$\theta = e^{-bt}\left[(A_1 + A_2)\,\omega s\sqrt{\omega_o^2 - b^2}\,t + i\,(A_1 - A_2)\sin\sqrt{\omega_o^2 - b^2}\,t\right]$$

we can write the above eqn. as

$$\theta = \theta_o e^{-bt}\sin(\omega t + \delta) \text{ ....(9.3.10)}$$

where $A_1+A_2 = \theta_o \sin\delta$; $i\,(A_1-A_2) = \theta_o\cos\delta$, $\omega = \sqrt{\omega_o^2 - b^2}$

and $i\,\sqrt{-\perp}$.

So the resultant motion will be an oscillatory motion with decresing time dependent amplitude and with of angular frequency $\omega = \sqrt{\omega_o^2 - b^2}$.

Now at t = o, θ =o, the above equation gives δ = o, so we have

$$\theta = \theta_o e^{-bt} \sin \omega t \quad \text{...(9.3.11)}$$

So the maximum deflection on either side of central position of coil will be at

$$t = \frac{T}{4}, \frac{3T}{4}, 5T/4$$

The variation of deflection with time is as show in the following fig. (9.3)

The successive maximum deflections are

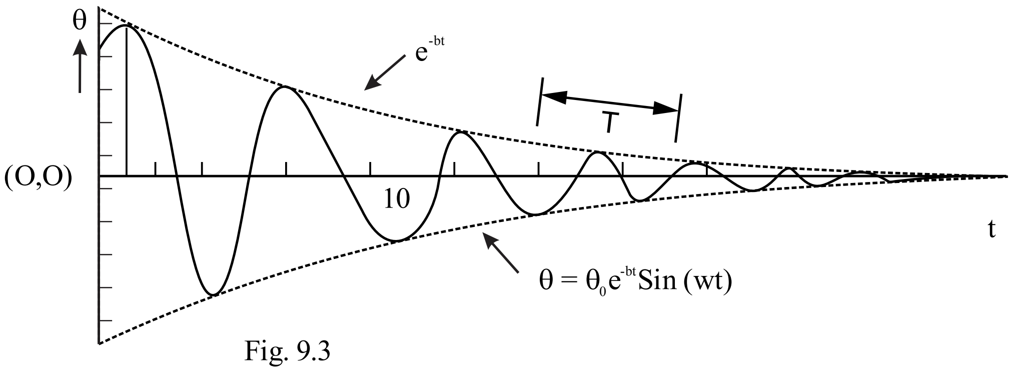


Fig. 9.3

$$\theta_1 = \theta_0 e^{-bT/4}, \ \theta_2 = \theta_0 e^{-b3T/4}, \ \theta_3 = \theta_0 e^{-b5T/4} \quad \text{...(9.3.12)}$$

The ratio of successive amplitude of deflections

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} \quad \frac{\theta_3}{\theta_4} \text{ ....} = e\frac{+bT}{2} = d.$$

d is known to be decrement per half cycle or simply decrement.

$l_n d = \lambda = bT/2$ is called logrithmic decrement.

Now from the eqn (9.3.12)

$$\theta_1 = \theta_0 e^{-bT/4} \text{ or } \theta_0 = \theta_1 e^{bT/4} = \theta_1 e^{\lambda/2}$$

As the damping factor is small of l is also very small.

So we can write $\theta_o = \theta_1 \,(1+\lambda/2)$

So the correct relation of change flowing and the first throw becomes

$$q = \frac{T}{2\pi}\ \frac{C}{nAB}\ \theta_1\ (1+ \lambda/2) \text{...}(9.3.13)$$

Procedure for Calculation of λ.

To find a resonable average value of λ, the first throw and the eleventh throw is noted them,

$$\theta_1/\theta_{11} = \frac{\theta_1}{\theta_2}\cdot\frac{\theta_2}{\theta_3}\cdot\frac{\theta_3}{\theta_4}\text{.......}\frac{\theta_{10}}{\theta_{11}} = e^{10bT/2} = e^{10\lambda}$$

or $$\lambda = \frac{1}{10}\left(en\left(\theta_1/\theta_{11}\right)\right)\text{..}(9.3.14)$$

Similarly two other sets of such readings are taken and the average value of λ is calculated.

Critical damping resistance (CDR)

We have already seen that the condition of ballastic galvanometer to be $b<\omega_0$, where b represents half the damping torque per unit moment of inertia (2b often reffered as damping factor) and ω_o is the angular velocity of the coil at t = 0, without damping.

Now $b = \frac{1}{2I}\ (a+\frac{G^2}{R})$, where R = R_G (galvanometer resistance) + R_e (External resistance with galvanometer circuit)

Thus we can write the condition for a galvanometer to be ballastic,

$$b = \frac{1}{2I}\ (a+\frac{G^2}{R}) < \omega_o = \sqrt{\frac{c}{I}} \Rightarrow R > \frac{G^2}{2\sqrt{Ic - a}}$$

or $R_e > \frac{G^2}{2\sqrt{Ic - a}} - R_G$ the limiting value of R_s.

$$R_e = G^2/(e\sqrt{Ic - a}\ ) - R_G \text{..}(9.3.15)$$

This value called CDR of the galvanometer. However the air damping factor due to air resistance is normally much less compared to the electromagnetic damping so we can neglect a and the CDR takes the form

$$Rc = \frac{G^2}{2\sqrt{Ic}} - R_G \quad \text{...(9.3.16)}$$

when the external resistance Re<Rc the damping force is sufficient to make the motion non-oscillatory and the galvanometer acts as dead-beat galvanometer.

Measurement of steady current

Now it the galvanometer circuit contains an external emt source E to supply a steady current i then the general equation of motion of galvanometer coil becomes

$$I\frac{d^2\theta}{dt^2} = - C\theta - (a+\frac{G^2}{R})\frac{d\theta}{dt} + niAB \quad \text{..(9.3.17)}$$

Then the resultant motion will be a super position of a damped harmonie motion about a steady deflection θ_s (say).

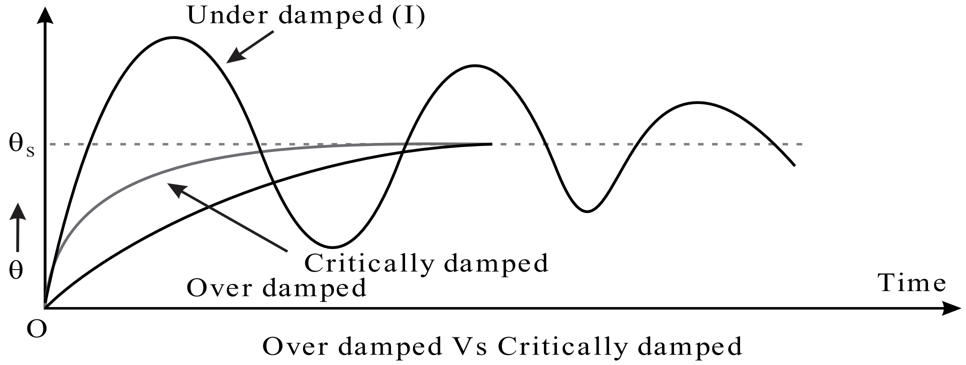


Fig. 9.4

Inthe mesurement of current we choose the external resistance less than the C.D.R so that the motion is over damped.

In this case $\quad ni_sAB = c\theta_s$ (θ_s = steady diflection for current is)

$$i_s = \frac{c}{nAB}\theta_c \quad \text{..(9.3.18)}$$

To increase the electro-magnetic damping the core of the dead beat galvanometer is made of soft iron which has large permiability.

Lamp and scale arrangement. (Fig. 9.5)

The deflection of coil is useally measured by using a lamp and scale arrangement. The arrangement consist of a lamp L mounted on a vertical stand ST. The lamp (collimates) a beam of light onthe mirror M attached to the suspension wire of galvanometer.

The reflected beam is received on a semi-transparent scale S held on same vertical stand ST and held horizontally and paralled to the mirror plane M. D is the distance between mirror and scale. It due to passage of current the coil reflects through θ and the deflection of light spot on scale is d, then tan2θ = d/D or θ = ½ d/D (when θ is very small)

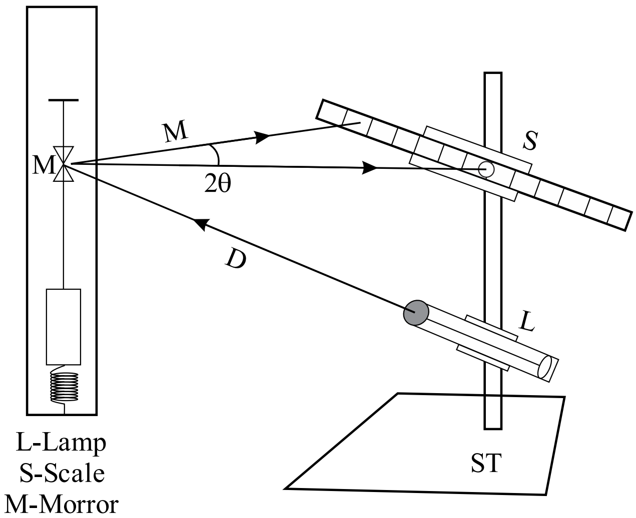


Fig. 9.5

Sensitivity of Galvanometer

The quality of a galvanometer to respond towards charge / current and voltage measurement is the measure of its sensilivity. Accordingly a galvanometer may have three types of sensitivity.

1. **Current Sensitivity.**

The current sensitivity of a galvanometer is the deflection in mm of the light spot on a scale placed 1m away from the galvanometer mirror intitially perpendicular mirror to the mirror palne of galvanometer due to the passage of 1μA current through the glavanometer.

So it a current of i_s μA produce a deflection d mm on the scale placed 1m away, then, the current sensititivity.

$$S_i = \frac{d}{i_s}\ \text{mm}/\mu\text{A}$$

2. **Voltage Sensitivity.**

The voltage sensitivity of a galvanometer is the deflection in mm of the light spot on a scale placed 1mm away from the galvanometer mirror, initially perpendicular to the mirror-plane of the galvanometer due to 1μV potential differnce across the galvanometer.

So if the potential differnce of Vs μV produced a diflection of dmm on the said scale then, the voltage sensitivity.

$$S_v = \frac{d}{V_s} \text{ mm}/\mu V$$

$$= d/i_s R_G$$

$$= \frac{Si}{R_G} \quad \text{...(9.3.19)}$$

3. Charge Sensitivity. (This is concerned to the ballastic galvanometer in practice)

The charge sensitivity of a galvanometer is the deflection in mm of a galvanometer is the away from the galvanometer mirror, initially perpendicular to the mirror plane of the galvanometer due to the passage of 1mc of charge through the galvanometer such that, the during the flow of charge the coil does not move.

So the charge sensitivity $Sq = \frac{d}{q}$ mm/μc.....................................(9.3.20)

Comparing the expression of q and i we have

$$Sq = \frac{T}{2\wedge} S_i \text{ (T = period of oscillation of galvanometer coil)...(9.3.21)}$$

9.4 Summary

We have learned the following lessons :

1. Basic principle of construction of ballastic and dead-beat galvanometer, differance between them.
2. Its operatinal physical parameter are
   1. Charge sensitivity

$$Sq = \frac{d\theta_0}{dq} = \frac{2\pi}{T}\,\frac{NAB}{C} = \frac{2\pi}{T}\,S_i$$

where S_i is the charge sensitivity

2. The current sensitivity S_i

$$S_i = \frac{d\theta}{d_i} = \frac{NAB}{C}$$

3. Voltage sensitivity (S_v)

$$S_V = \frac{d\theta}{dv} = \frac{1}{Rg}\frac{d\theta}{di} = \frac{S_i}{Rg}$$

As $S_i \propto \sqrt{Rg}$ therefore $Sv \propto \frac{1}{\sqrt{R_g}}$

4. CDR is given by

$$R_C = \frac{G^2}{2\sqrt{IC}} - Rg$$

9.5 Review Questions and Answer

1. Plot a neat diagram of Ballastic galvanometer write the names of its various components.

Ans : See the text

2. Give the theory of moving coil galvanometer. Explain the conditions under which the galvanometer works a) ballastic b) dead-beat.

Ans : See the text

3. The 1st and 11th throw of a ballastic galvanometer are 5cm and 12cm respectively. Calculate the value of logarithmic decrement.

   So in $\lambda = \frac{1}{10}\, ln \frac{\theta_1}{\theta_{11}}$

4. Define charge sensitivity, current sensitivity and voltage sensitivity of ballastic galvanometer.

Ans : See Article 9.3

5. what are difference between ballastic and Dead-beat Galvanometer?

Ans :

Ballastic	Deat-beat
1. Damping is small and the motion is oscillatory.	1. Damping is large and the motion is non-oscillatory.
2. It measure charge	2. It measure current
3. The transient flow of charge causes an impluse while the coil has not moved sufficently from its rest position. This condition is achieved by enhancing the movement of inertia of the coil to have larges time period of oscillation to about 10-20 seconds. The external driving torque is zero when the coil rotates	3. The coil rotates under the action of torque.
4. The coil frame is non-metalic to reduce electromagnetic damping.	4. The coil is around on a metalic frame to increase electromagnetic damping.
5. The ballastic throw measure the charge.	5. The steady deflection measure the current.
6. The external resistance of the glavanometer circuit must be greater than CDR to ensure oscillatory motion.	6. The total external resistance must be less than CDR to obtain non-oscillatory motion.(aperiodic)

9.6 Problems and Solution

Q. 1. A moving coil galvanometer has the following characteristics _ number of turns of the coil = 50; Area of coil = 70 mm^2; Resistance of coil = 30Ω; tlux density of redial field = O.I.T. Torsional constant of the suspension wire = 7 x 10^8Nm/rad. Calculate the current and voltage sensitivity.

Solution:

Given N=50, A = 70mm^2, B = O.I.T, C = 7 x 10^8 Nm/rad, R = 30Ω

Current sensitivity (Is) $= \dfrac{NAB}{C} = \dfrac{50 \times 70 \times 10^{-6} \times 0.1 \times 10^{+8}}{7}$

$= \dfrac{350 \times 100}{7}$

= 5 x 10 div / amp

= 5div / mA.

Voltage sensitivity (Vs) $= \dfrac{NAB}{C\ \ R} = \dfrac{Is}{R}$

$= \dfrac{5}{30} = \dfrac{6}{6}$

= 0.167 div/mV.

Q.2. What is galvanometer constant?

Solution :

In a moving coil galvanometer the current (I) passing through the galvanometer is directly propertional to its deflection(θ).

$i = G\theta$

where $G = \dfrac{C}{NAB}$ = galvanometer constant.

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